Applied Mathematics and Mechanics (English Edition), 2006, **27**(8):1149–1156

SPATIO-TEMPORAL CHAOTIC SYNCHRONIZATION FOR MODES COUPLED TWO GINZBURG-LANDAU EQUATIONS *

HU Man-feng (胡满峰), XU Zhen-yuan (徐振源)

(School of Science, Southern Yangtze University, Wuxi 214122, Jiangsu Province, P. R. China)

(Communicated by LIU Zeng-rong)

Abstract: On the basis of numerical computation, the conditions of the modes coupling are proposed, and the high-frequency modes are coupled, but the low frequency modes are uncoupled. It is proved that there exist an absorbing set and a global finite dimensional attractor which is compact and connected in the function space for the high-frequency modes coupled two Ginzburg-Landau equations(MGLE). The trajectory of driver equation may be spatio-temporal chaotic. One associates with MGLE, a truncated form of the equations. The prepared equations persist in long time dynamical behavior of MGLE. MGLE possess the squeezing properties under some conditions. It is proved that the complete spatio-temporal chaotic synchronization for MGLE can occur. Synchronization phenomenon of infinite dimensional dynamical system (IFDDS) is illustrated on the mathematical theory qualitatively. The method is different from Liapunov function methods and approximate linear methods.

Key words: complete synchronization; Ginzberg-Landau equations; attractor; spatio-temporal chaos

Chinese Library Classification: O175.29
2000 Mathematics Subject Classification: 35B99
Digital Object Identifier(DOI): 10.1007/s 10483-006-0816-y

Introduction

Synchronization phenomenon is of fundamental importance in telecommunication, electronic circuits, nonlinear optics and biological systems. The phenomenon has been studied extensively by theoretical, numerical and experimental means[1−3]. It is believed that an in-depth study and understanding of synchronization will greatly benefit the advancement of science and technology. Recently, synchronization and control of infinite dimensional dynamical system (IFDDS) have received great attention[4−7]. The synchronization of finite dimensional dynamical system (FDDS) had many theoretical results[1−3], but much of the present understanding on synchronization of IFDDS was achieved with the aid of numerical computation. It has not been seen the complete theoretical results. In Ref.[5] the synchronized chaos in geophysical fluid dynamics systems has been investigated. The coupling method was the following:

$$F_k^A = \mu_k^c[q_k^A - q_k^B] + [\mu_0 - \mu_k^c][q_k^* - q_k^A],$$

where the flow (the solution q of the equation) has been decomposed spectrally and the subscript k on each quantity indicates the wave number, q_k indicates spectral component. If the coupling coefficient μ_k^c is allowed to vary with the wave number, then we can arrange to couple the small-scale/high-frequency components while leaving the large-scale modes/low frequency

* Received Aug.17, 2004; Revised Feb.24, 2006
Project supported by the National Natural Science Foundation of China (No.10372054)
Corresponding author XU Zhen-yuan, Professor, E-mail: xu_zhenyuan@yahoo.com.cn

components uncoupled, by setting

$$\begin{cases} \mu_k^c = 0, & \text{if} \quad |k_x| \le k_{x_0}, \quad |k_y| \le k_{y_0}, \\ \mu_k^c = \mu_0[1 - (\frac{k_0}{k})^4], & \text{otherwise.} \end{cases}$$

The coupling is called as modes high-frequency coupling. In Ref.[8] the occurrence of frequency synchronization, phase synchronization, generalized synchronization of a pair of unidirectionally weakly coupled nonidentical Ginzberg-Landau equations was demonstrated. As everyone known, the problem for synchronization of IFDDS is an interesting and difficult problem. The main purpose of this paper is to investigate the spatio-temporal synchronization of two identical Ginzburg-Landau equations under modes high-frequency coupling from different initial conditions. In the 90s of the 20th century, Foias and Temam[8] had established a theory for global attractor and inertial manifolds, and the theory was applied to a type of IFDDS systems successfully, for example, Navier-Stokes equations, Ginzberg-Landau equations. In 1996, Li and Wiggins[10] had investigated the following perturbed nonlinear Schrödinger equations:

$$\mathrm{i}q_t = q_{xx} + 2[q\bar{q} - \omega^2]q + \mathrm{i}\varepsilon[\hat{D}q - \Gamma], \tag{1}$$

and proved the existence of a symmetric pair of nontrivial homoclinic orbits, which corresponds to the existence of spatio-temporal chaos. Let us consider the following Ginzburg-Landau equation:

$$q_{1t} + (\mathrm{i} + \varepsilon\beta)q_{1xx} + (\varepsilon k + 2\mathrm{i})|q_1|^2 q_1 + (\varepsilon\alpha - 2\omega^2\mathrm{i})q_1 + \varepsilon\Gamma = 0, \tag{2}$$

where $-\varepsilon\beta = \lambda > 0, \varepsilon k > 0, \varepsilon\Gamma > 0$, $\Omega = [0, L]$. q_1 is Ω periodic function and

$$q_1(x, 0) = q_{10}(x), \qquad x \in \Omega. \tag{3}$$

We can prove that there exists a global attractor which is compact, connected and finite dimensional on the basis of the theory of Ref.[9], also can prove that there exists a pair of nontrivial homoclinic orbits which corresponds to the existence of spatio-temporal chaos on the basis of the theory of Ref.[10].

In this paper we investigate the complete spatio-temporal chaos synchronization for two identical Ginzberg-Landau equations under modes high-frequency coupling. The coupling equation with Eq.(2) is

$$q_{2t} + (\mathrm{i} + \varepsilon\beta)q_{2xx} + (\varepsilon k + 2\mathrm{i})|q_2|^2 q_2 + (\varepsilon\alpha - 2\omega^2\mathrm{i})q_2 + \varepsilon\Gamma = F(q_1, q_2), \tag{4}$$

where q_2 is also Ω periodic function and

$$q_2(x, 0) = q_{20}(x), \qquad x \in \Omega. \tag{5}$$

F is modes coupling function defined as $F_k = \mu_k(q_k^1 - q_k^2), k = 1, 2, \cdots$, where q_k^1, q_k^2 are the projectors of q_1, q_2 onto the space spanned by W_k, where W_k is an orthonormal eigenvector of $A = -\triangle = -\frac{\partial^2}{\partial x^2}$. Suppose μ_k is a bounded monotone increasing function:

$$\begin{cases} \mu_{k_0} \le \mu_k \le \mu_0, & \text{when } 1 \le k_0 \le k, \\ \mu_k = 0, & \text{when } k < k_0. \end{cases}$$

Definition *Ginzburg-Landau equations* (2) − (5) *are called as the complete synchronization, if*

$$\lim_{t \longrightarrow \infty} |q_1(x, t) - q_2(x, t)| = 0,$$

where $|\cdot|$ *indicates the norm in the function space.*

Oftentimes Liapunov function methods and approximate linear methods are adopted to study the synchronization of FDDS, but it is difficult to use them to study IFDDS. In this paper, we investigate successfully the synchronization of IFDDS using the theory[9] of absorbing sets, attractor and inertial manifolds.

The rest of this paper is organized as follows. In Section 1, we present the mathematical setting for Eqs.(2)–(5) and show the existence of an absorbing set and an attractor. In Section 2, we prepare Eqs.(2) and (4), but it don't affect long time dynamical behavior, and prove the squeezing properties. In Section 3, we prove that in the modes high-frequency coupling Eqs.(2)–(5) can realize the synchronization under some conditions.

1 Absorbing Set and Attractor

For the in-depth investigation of Eqs.(2)–(5), we introduce complex Sobolev space. In general, we denote by $\vec{X},\vec{Y}$ the complex space of function space X,Y. For example $\vec{L}^2(\Omega)$ is the complex space of $L^2(\Omega)$. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the scalar product and the norm in $L^2(\Omega)$ and $\vec{L}^2(\Omega)$, respectively. we write $H=L^2(\Omega), V=\vec{H}^1_{\rm per}(\Omega)$. $||\cdot||$ denotes the norm of V. In the case, $Au=-\triangle u$, we denote by W_j,λ_j the eigenvectors (orthonormal in H) and eigenvalues of A in H, respectively,

$$AW_j=\lambda_jW_j,\quad j\geq 1;\quad 0\leq\lambda_1<\lambda_2<\cdots<\lambda_j<\cdots,\ \lambda_j\to\infty\ (j\to\infty). \tag{6}$$

For some integer N, we consider the operators P_N and $Q_N=I-P_N$, where P_N is the projector from H onto span$\{W_1,W_2,\cdots,W_N\}$. Problem (2)–(5) is now equivalent to the following functional evolution equations:

$$\frac{dq_1}{dt}+(-{\rm i}+\lambda)Aq_1+(\varepsilon k+2{\rm i})|q_1|q_1+(\varepsilon\alpha-2\omega^2{\rm i})q_1+\varepsilon\Gamma=0, \tag{7}$$

$$\frac{dq_2}{dt}+(-{\rm i}+\lambda)Aq_2+(\varepsilon k+2{\rm i})|q_2|q_2+(\varepsilon\alpha-2\omega^2{\rm i})q_2+\varepsilon\Gamma=F(q_1,q_2), \tag{8}$$

$$q_j(0)=q_{j0},\qquad j=1,2.$$

The existence and uniqueness of solutions are sufficient to define the semigroup: $S_j(t):q_{j0}\to q_j(t)$.

We will prove the existence of an absorbing set in H for Eqs.(7) and (8). First, the existence of an absorbing set for Eq.(7) was proved in Ref.[9]. For the case $r=-(\varepsilon\alpha-\frac{\varepsilon\Gamma}{2})$ and $r>0$, we have

$$|q_1(t)|^2\leq|q_{10}|^2\exp(-2rt)+\frac{1}{2r}\left[\frac{4r^2}{\varepsilon k}+\varepsilon\Gamma\right]|\Omega|(1-\exp(-2rt)), \tag{9}$$

$$\lim_{t\longrightarrow\infty}\sup|q_1(t)|^2\leq\rho_{10}^2,\quad \rho_{10}^2=\frac{1}{2r}\left[\frac{4r^2}{\varepsilon k}+\varepsilon\Gamma\right]|\Omega|. \tag{10}$$

There exists an absorbing set $B_0=B_H(0,\hat{\rho}_{10})$ (any ball of H centered at 0 of radius $\hat{\rho}_{10}>\rho_{10}$) which is positively invariant for the semigroup $S_1(t)$. If B is a bounded set of H, included in a ball $B(0,R)$ of H centered at 0 of radius R. When $t\geq t_0=t_0(B,B_0), t_0=\frac{1}{2r}\lg\frac{R^2}{(\hat{\rho}_{10})^2-(\rho_{10})^2}$, $S_1(t)B\subset B_0$. Integrating in time t from t to $t+\tilde{r}(\tilde{r}>0)$ and supposing $q_{10}\in B, t\geq t_0$, we obtain

$$\int_t^{t+\tilde{r}}\{2\lambda||q_1||^2+\varepsilon k|q_1|^4_{L^4}+2r|q_1|^2\}ds\leq\hat{\rho}_{10}^2+\frac{4r^2\tilde{r}}{\varepsilon k}|\Omega|+\varepsilon\Gamma|\Omega|\tilde{r}. \tag{11}$$

We also consider Eq.(8), analogous to Eq.(7), and obtain

$$\frac{1}{2}\frac{d}{dt}|q_2|^2+\lambda||q_2||^2+\varepsilon k|q_2|^4_{L^4}+\varepsilon\alpha|q_2|^2+\varepsilon\Gamma\int_\Omega{\rm Re}\bar{q}_2dx=\int_\Omega F(q_1,q_2)\bar{q}_2dx.$$

Only considering the case $r+h>0, h=\frac{\mu_0-2\mu_{k_0}}{2}$,

$$\begin{aligned}&\frac{\varepsilon k s^4}{2}-2(r+h)s^2\geq-\frac{2}{\varepsilon k}(r+h)^2,\\&\frac{d}{dt}|q_2|^2+2\lambda||q_2||^2+\varepsilon k|q_2|^4_{L^4}+2(r+h)|q_2|^2\leq\frac{4(r+h)^2}{\varepsilon k}|\Omega|+\varepsilon\Gamma|\Omega|+\mu_0\hat{\rho}^2_{10}.\end{aligned}\tag{12}$$

Using the Gronwall lemma we see that

$$\begin{cases}|q_2(t)|^2\leq|q_{20}|^2\exp(-2r-2h)t+\dfrac{4(r+h)^2|\Omega|+(\varepsilon\Gamma|\Omega|+\mu_0\hat{\rho}^2_{10})\varepsilon k}{2(r+h)\varepsilon k}(1-\exp(-2r-2h)t),\\ \lim\limits_{t\longrightarrow\infty}\sup|q_2(t)|^2\leq\rho^2_{20},\quad \rho^2_{20}=\dfrac{4(r+h)^2|\Omega|+(\varepsilon\Gamma|\Omega|+\mu_0\hat{\rho}^2_{10})\varepsilon k}{2(r+h)\varepsilon k}.\end{cases}\tag{13}$$

There exists an absorbing set $B_0^{'}=B_H(0,\hat{\rho}_{20})$(any ball of H centered at 0 of radius $\hat{\rho}_{20}>\rho_{20}$) which is positively invariant for the semigroup $S_2(t)$. If B is a bounded set of H, included in a ball $B(0,R)$ of H centered at 0 of radius R. When $t\geq t_0=t_0(B,B_0^{'})$ and $t_0=\frac{1}{2r}\lg\frac{R^2}{(\hat{\rho}_{20})^2-(\rho_{20})^2}$, $S_2(t)B\subset B_0^{'}$. Integrating in time t from t to $t+\tilde{r}(\tilde{r}>0)$ and supposing $q_{20}\in B, t\geq t_0$ we obtain

$$\int_t^{t+\tilde{r}}\{2\lambda||q_2||^2+\varepsilon k|q_2|^4_{L^4}+2r|q_2|^2\}ds\leq\hat{\rho}^2_{20}+\frac{4(r+h)^2\tilde{r}}{\varepsilon k}|\Omega|+\varepsilon\Gamma|\Omega|\tilde{r}+\mu_0\hat{\rho}^2_{10}\tilde{r}.\tag{14}$$

We now prove the existence of an absorbing set for Eqs.(2)–(5) in V. The existence of an absorbing set in V for Eq.(7) was proved in Ref.[9]. we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||q_1||^2+\lambda|\Delta q_1|^2+\varepsilon\alpha||q_1||^2-\varepsilon\Gamma\int_\Omega\mathrm{Re}\Delta\bar{q}_1dx\\=&\mathrm{Re}(\varepsilon k+2\mathrm{i})\int|q_1|^2q_1\Delta\bar{q}_1\\\leq&3[(\varepsilon k)^2+4]|q_1|^4_{L^4}|\Delta q_1|^4_{L^4}+\frac{|\Omega|(\varepsilon k)^2}{2\lambda}+\frac{\lambda}{2}|\Delta q_1|^2,\end{aligned}$$

then

$$\frac{d}{dt}||q_1||^2\leq2(-\varepsilon\alpha+C_3^{'}|q_1|^4_{L^4})||q_1||^2+\frac{|\Omega|}{\lambda}(\varepsilon\lambda)^2.$$

At this point we apply the uniform Gronwall lemma with

$$\begin{aligned}&y=||q_1||^2,\quad g=2(-\varepsilon\alpha+C_3^{'}|q_1|^4_{L^4}),\quad \tilde{h}=\frac{|\Omega|}{\lambda}(\varepsilon\Gamma)^2,\\&\int_t^{t+\tilde{r}}g(s)ds\leq a_1,\quad\int_t^{t+\tilde{r}}\tilde{h}(s)ds\leq a_2,\quad\int_t^{t+\tilde{r}}y(s)ds\leq a_3,\end{aligned}$$

then we can obtain

$$||q_1(t)||^2\leq\left(\frac{a_3}{\tilde{r}}+a_2\right)\exp a_1\qquad(t\geq t_0+\tilde{r}),\tag{15}$$

where $\tilde{r}>0$ is arbitrarily chosen, $q_{10}\in B$. If B is a bounded set of V, then it is also a bounded set of H. Then $S(t)B\subset B_0$ $(t\geq t_0(B,B_0)); S(t)B\subset B_1$ $(t_0\geq t_0+\tilde{r})$, where B_1 is the ball of V centered at 0 of radius ρ_1,

$$\rho_1^2=\hat{\rho}^2_{10}+\left(\frac{a_3}{\tilde{r}}+a_2\right)\exp a_1.\tag{16}$$

Thus B_1 is an absorbing set in V for $S_1(t)$. Analogous to Eq.(7), multiplying Eq.(8) by $-\Delta\bar{q}_2$ and integrating over Ω, using the uniform Gronwall lemma, we have

$$||q_2(t)||^2 \leq \left(\frac{a_3^{'}}{\tilde{r}} + a_2^{'}\right) \exp a_1^{'} \qquad (t \geq t_0 + \tilde{r}), \tag{17}$$

where $a_j^{'}, j = 1, 2, 3$ are analogous to $a_j, j = 1, 2, 3$ of Eq.(15). Thus we obtain the existence of an absorbing set $B_2 = B_V(0, \rho_2)$ in V for $S_2(t)$. Since the injection from V to H is compact, we have

Theorem 1 *We consider the dynamical system associated with the two Ginzburg-Laudau equations* (2)–(5) *supplement by periodic boundary conditions under modes coupling with* $\lambda = -\varepsilon\beta > 0, \varepsilon k > 0$. *This dynamical system possesses an attractor* $\Im$*, which is compact, connected and maximal in* $\vec{L}^2(\Omega) \times \vec{L}^2(\Omega)$.

2 Prepared Equations and Squeezing Properties

In order to overcome the difficulties related to the behavior of nonlinear term for large of $|q_1|$ and $|q_2|$, we associate with Eqs.(7), (8) a truncated form of the equations, called the prepared equations, and that we will define. The prepared equations will persist in long time dynamical behavior of Eqs.(2)–(5). We choose $\rho > 0$ such that absorbing sets B_1, B_2 (and hence $\Im$) are included in the ball of $D(A^\alpha)$ centered at 0 of radius $\frac{\rho}{2}$. We also choose a C^∞ function θ:

$$\theta(s) = 1, \quad 0 \leq s \leq 1; \quad \theta(s) = 0, \quad s \geq 2; \quad \sup_{s\geq 0} |\theta^{'}(s)| \leq 2.$$

Let

$$\theta_\rho(s) = \theta\left(\frac{s}{\rho}\right), \quad B(u,u) = (\varepsilon k + 2\mathrm{i})|u|^2 u, \quad B_\theta(u,u) = \theta_\rho(|A^\alpha u|)(\varepsilon k + 2\mathrm{i})|u|^2 u.$$

Then the prepared equations of Eqs.(7) and (8) are

$$\frac{dq_1}{dt} + (-\mathrm{i} + \lambda)Aq_1 + (\varepsilon k + 2\mathrm{i})B_\theta(q_1, q_1) + (\varepsilon\alpha - 2\omega^2\mathrm{i})q_1 + \varepsilon\Gamma = 0, \tag{18}$$

$$\frac{dq_2}{dt} + (-\mathrm{i} + \lambda)Aq_2 + (\varepsilon k + 2\mathrm{i})B_\theta(q_2, q_2) + (\varepsilon\alpha - 2\omega^2\mathrm{i})q_2 + \varepsilon\Gamma = F(q_1, q_2). \tag{19}$$

For $\frac{1}{8} \leq \alpha \leq \frac{3}{8}$, the following lemmas are proved easily:

Lemma 1 *B_θ is a globally bounded operator from $D(A^\alpha)$ to $D(A^{\alpha-\frac{1}{2}})$.*

$$\sup_{u\in D(A^\alpha)} |A^{\alpha-\frac{1}{2}} B_\theta(u,u)| \leq M_1. \tag{20}$$

Lemma 2 *B_θ is a globally Lipschitze mapping from $D(A^\alpha)$ to $D(A^{\alpha-\frac{1}{2}})$.*

$$|\theta_\rho(|A^\alpha u_1|)A^{\alpha-\frac{1}{2}}B(u_1,u_1) - \theta_\rho(|A^\alpha u_2|)A^{\alpha-\frac{1}{2}}B(u_2,u_2)| \leq M_2|A^\alpha(u_1 - u_2)|. \tag{21}$$

We now prove the squeezing properties. From Eqs.(18) and (19) we obtain

$$\frac{dP_N q_1}{dt} + (-\mathrm{i} + \lambda)AP_N q_1 + P_N(\varepsilon k + 2\mathrm{i})B_\theta(q_1, q_1) + (\varepsilon k - 2\omega^2\mathrm{i})P_N q_1 + \varepsilon\Gamma = 0,$$

$$\frac{dP_N q_2}{dt} + (-\mathrm{i} + \lambda)AP_N q_2 + P_N(\varepsilon k + 2\mathrm{i})B_\theta(q_2, q_2) + (\varepsilon k - 2\omega^2\mathrm{i})P_N q_2 + \varepsilon\Gamma = P_N F(q_1, q_2).$$

Setting

$$N \le k_0 - 1, \quad P = P_N(q_1 - q_2), \quad R = \theta_N(q_1 - q_2), \quad q = q_1 - q_2,$$

then we have

$$\frac{dP}{dt} + (-\mathrm{i} + \lambda)AP + P_N(\varepsilon k + 2\mathrm{i})(B_\theta(q_1, q_1) - B_\theta(q_2, q_2)) + (\varepsilon k - 2\omega^2 \mathrm{i})P = -P_N F(q_1, q_2). \quad (22)$$

We multiply Eq.(22) by $A^{2\alpha}\bar{P}$ and integrate it over Ω, take the real part of the equation, then we obtain

$$\begin{aligned}\frac{1}{2}\frac{d|A^\alpha P|^2}{dt} + \lambda|A^{\alpha+\frac{1}{2}}P|^2 \ge & -[(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[|A^\alpha P| + |A^\alpha R|]|A^{\alpha+\frac{1}{2}}P| \\ & - (\varepsilon\alpha + \mu_N)|A^\alpha P|^2. \end{aligned} \quad (23)$$

Similar to those used in the proofs of Eq.(23), we have

$$\begin{aligned}&\frac{dR}{dt} + (-\mathrm{i} + \lambda)AR + Q_N(\varepsilon k + 2\mathrm{i})(B_\theta(q_1, q_1) - B_\theta(q_2, q_2)) + (\varepsilon k - 2\omega^2 \mathrm{i})R = -Q_N F(q_1, q_2), \\ &\frac{1}{2}\frac{d|A^\alpha R|^2}{dt} + \lambda|A^{\alpha+\frac{1}{2}}R|^2 \le -[(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[|A^\alpha P| + |A^\alpha R|]|A^{\alpha+\frac{1}{2}}P| \\ &\qquad - (\varepsilon\alpha + \mu_{N+1})|A^\alpha P|^2. \end{aligned} \quad (24)$$

We denote by Σ the cone of $V \times V$,

$$\Sigma = \{v \in D(A^\alpha), |A^\alpha Q_N v| \le \mu|A^\alpha P_N v|, \mu > 0\}.$$

If $A^\alpha R \in \partial\Sigma$, then

$$|A^\alpha R| = \mu|A^\alpha P|, \quad f(z) = -\lambda z^2 + [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2(|A^\alpha P| + |A^\alpha Q|)z$$

is quadratic function. For $z > \frac{1}{2\lambda}[(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2(|A^\alpha P| + \mu|A^\alpha P|)$, $f(z)$ is decreasing. When λ_{N+1} is sufficiently big, we have

$$\begin{aligned}|A^{\alpha+\frac{1}{2}}R| \ge & \lambda_{N+1}^{\frac{1}{2}}|A^\alpha R| = \lambda_{N+1}^{\frac{1}{2}}\mu|A^\alpha P| \\ \ge & \frac{1}{2\lambda}[(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2(1 + \mu)|A^\alpha P|.\end{aligned}$$

Equation (24) becomes

$$\begin{aligned}\frac{1}{2}\frac{d|A^\alpha R|^2}{dt} \le & -\lambda\lambda_{N+1}|A^\alpha R|^2 + [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[1 + \frac{1}{\mu}]|A^\alpha R|^2 \lambda_{N+1}^{\frac{1}{2}} \\ & - (\varepsilon\alpha + \mu_{N+1})|A^\alpha R|^2.\end{aligned}$$

Considering

$$\begin{aligned}\frac{1}{2}\frac{d(|A^\alpha R|^2 - \mu^2|A^\alpha P|^2)}{dt} \le & |A^\alpha R|^2\{-\lambda\lambda_{N+1} + [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[1 + \frac{1}{\mu}]\lambda_{N+1}^{\frac{1}{2}} - (\varepsilon\alpha + \mu_{N+1}) \\ & + \lambda\lambda_N + [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[1 + \mu]\lambda_N^{\frac{1}{2}} + \varepsilon\alpha + \mu_N\},\end{aligned}$$

when

$$\begin{aligned}\lambda\lambda_{N+1} + \mu_{N+1} > & [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[1 + \frac{1}{\mu}]\lambda_{N+1}^{\frac{1}{2}} + \lambda\lambda_N \\ & + [(\varepsilon k)^2 + 4]^{\frac{1}{2}} M_2[1 + \mu]\lambda_N^{\frac{1}{2}} + \mu_N,\end{aligned} \quad (25)$$

we have

$$\frac{1}{2}\frac{d(|A^\alpha R|^2-\mu^2|A^\alpha P|^2)}{dt}\leq 0.$$

This shows that if $A^\alpha q(t_1)\in\Sigma$, then $A^\alpha q = A^\alpha(q_1-q_2)$ cannot leave Σ at $t>t_1$. If $A^\alpha q$ is not in Σ, then $|A^\alpha R|\geq\mu|A^\alpha P|$, we have

$$\frac{1}{2}\frac{d|A^\alpha R|^2}{dt}\leq -|A^\alpha R|^2\{\lambda\lambda_{N+1}-[(\varepsilon k)^2+4]^{\frac{1}{2}}M_2[1+\frac{1}{\mu}]\lambda_{N+1}^{\frac{1}{2}}+\varepsilon\alpha+\mu_{N+1}\}.$$

Letting

$$\xi=\lambda\lambda_{N+1}-[(\varepsilon k)^2+4]^{\frac{1}{2}}M_2[1+\frac{1}{\mu}]\lambda_{N+1}^{\frac{1}{2}}+\varepsilon\alpha+\mu_{N+1}>0, \tag{26}$$

we obtain

$$|A^\alpha R|^2\leq|A^\alpha R(0)|^2\exp(-2\xi t). \tag{27}$$

For $t\to\infty$, $|A^\alpha R|\to 0$. Finally we obtain the following theorem (the squeezing property):

Theorem 2 *If Eqs.*(25) *and* (26) *are satisfied for Eqs.*(7) *and* (8) *supplemented periodic boundary condition, then*

(i) *if* $A^\alpha q_1(0)\in A^\alpha q_2(0)+\Sigma$, *then* $A^\alpha q_1(t)\in A^\alpha q_2(t)+\Sigma,\forall t\geq 0$;

(ii) *if* $A^\alpha q_1(0)$ *is not in* $A^\alpha q_2(0)+\Sigma$, *then either* $A^\alpha q_1(t_0)\in A^\alpha q_2(t_0)+\Sigma$ *for some* $t_0>0$, *consequently* $A^\alpha q_1(t)\in A^\alpha q_2(t)+\Sigma$ *for all* $t\geq t_0$; *or* $A^\alpha q_1(t)$ *is not in* $A^\alpha q_2(t)+\Sigma$, *then* $A^\alpha Q_N q_1(t)-A^\alpha Q_N q_2(t)$ *decays exponentially as* $t\longrightarrow\infty$.

3 Synchronization for Modes Coupling

When $A^\alpha q$ is out of the cone Σ, *i.e.*, $|A^\alpha R|\geq\mu|A^\alpha P|$, for $t\to\infty$, $|A^\alpha R|\to 0$, $|A^\alpha P|\to 0$, then $|A^\alpha(q_1-q_2)|\to 0$. When $A^\alpha q$ is in the cone Σ, *i.e.*, $|A^\alpha R|\leq\mu|A^\alpha P|$. We modify Eqs.(20) and (21) and obtain

Lemma 3 *In the cone* Σ, B_θ *is a globally bounded operator from* $D(A^\alpha)$ *to* $D(A^{\alpha-\frac{1}{2}})$.

$$\sup_{u\in D(A^\alpha)}|A^{\alpha-\frac{1}{2}}B_\theta(u,u)|\leq\tilde{M}_1\leq M_1.$$

Lemma 4 *In the cone* Σ, B_θ *is a globally Lipschitze mapping from* $D(A^\alpha)$ *to* $D(A^{\alpha-\frac{1}{2}})$.

$$|\theta_\rho(|A^\alpha u_1|)A^{\alpha-\frac{1}{2}}B(u_1,u_1)-\theta_\rho(|A^\alpha u_2|)A^{\alpha-\frac{1}{2}}B(u_2,u_2)|\leq\tilde{M}_2|A^\alpha(u_1-u_2)|,$$

where $\tilde{M}_2\leq M_2$.

We obtain that

$$\frac{1}{2}\frac{d|A^\alpha P|^2}{dt}+\lambda|A^{\alpha+\frac{1}{2}}P|^2\leq[(\varepsilon k)^2+4]^{\frac{1}{2}}\tilde{M}_2[|A^\alpha P|+|A^\alpha R|]|A^{\alpha+\frac{1}{2}}P|-(\varepsilon\alpha+\mu_{k_0})|A^\alpha P|^2.$$

$f(z)=-\lambda z^2+[(\varepsilon k)^2+4]^{\frac{1}{2}}\tilde{M}_2(|A^\alpha P|+|A^\alpha Q|)z$ is quadratic function. When $z<\frac{1}{2\lambda}[(\varepsilon k)^2+4]^{\frac{1}{2}}\tilde{M}_2(|A^\alpha P+\mu|A^\alpha P|)$, $f(z)$ is increasing. When λ_N is appropriately selected,

$$|A^{\alpha+\frac{1}{2}}P|\leq\lambda_N^{\frac{1}{2}}|A^\alpha P|\leq\frac{1}{2\lambda}[(\varepsilon k)^2+4]^{\frac{1}{2}}\tilde{M}_2(1+\mu)|A^\alpha P|, \tag{28}$$

$$\frac{1}{2}\frac{d|A^\alpha P|^2}{dt}\leq -\{\lambda\lambda_N-[(\varepsilon k)^2+4]^{\frac{1}{2}}\tilde{M}_2[1+\mu]\lambda_N^{\frac{1}{2}}+\varepsilon\alpha+\mu_{k_0}\}|A^\alpha P|^2.$$

Letting

$$\eta = \lambda\lambda_N - [(\varepsilon k)^2 + 4]^{\frac{1}{2}} \tilde{M}_2 [1 + \mu] \lambda_N^{\frac{1}{2}} + \varepsilon\alpha + \mu_{k_0} > 0, \tag{29}$$

we obtain

$$|A^\alpha P(t)|^2 \le |A^\alpha P(0)|^2 \exp(-2\eta t).$$

We obtain that when $t \to \infty$, $|A^\alpha P(t)| \to 0$, then $|A^\alpha R(t)| \to 0$, *i.e.*, $|A^\alpha (q_1 - q_2)| \to 0$. Finally we have:

Theorem 3 *If Eqs.*(28), (29), (25) *and* (26) *are satisfied, Ginzburg-Landau equations* (2)−(5) *can realize the synchronization under modes coupling.*

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