

NONLINEAR NORMAL MODES AND THEIR SUPERPOSITION IN A TWO DEGREES OF FREEDOM ASYMMETRIC SYSTEM WITH CUBIC NONLINEARITIES*

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Abstract

This paper investigates nonlinear normal modes and their superposition in a two degrees of freedom asymmetric system with cubic nonlinearities for all nonsingular conditions, based on the invariant subspace in nonlinear normal modes for the nonlinear equations of motion. The focus of attention is to consider relation between the validity of superposition and the static bifurcation of modal dynamics. The numerical results show that the validity has something to do not only with its local restriction, but also with the static bifurcation of modal dynamics.

Key words nonlinear normal mode, asymmetric system, nonlinear vibration, nonlinear dynamics

I. Introduction

The modal analysis for studying the linear systems has played an important role and many scholars have been trying to apply the method for studying nonlinear vibration systems for many years. In a series of papers, Rosenberg^[1-5] introduced the concept of nonlinear normal modes. According to Rosenberg, free vibrations in normal modes are vibration-in-unison. In a recent work, Shaw^[6] associates this view with the invariant manifold in the dynamical systems and points out that the subspace of nonlinear normal modes is also an invariant set. Meanwhile, he provides a constructive method to find nonlinear normal modes.

Besides the above, many methods have been used to study nonlinear normal modes. Anand^[7] analysed the free vibration of a system of the massed connected by means of nonlinear springs. He used one-term Fourier approximations for the periodic motions. In [8], group representation theory was used to investigate normal modes of symmetric systems. Vakakis^[9] introduced similar normal modes and their bifurcations by means of balancing diagrams. A numerical study of bifurcating normal modes was given in [10].

Nearly all previous work on nonlinear normal modes deals with similar ones and the chosen physical systems are symmetric ones. Although Shaw provides a new studying field for nonlinear normal modes, a basic problem has not been investigated in details, that is, how many degrees the superposed solution of nonlinear normal modes can describe solution of the

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original system if it has two or more normal modes. For it, a two degree of freedom system of free vibration with cubic nonlinearities is considered as follows

$$\left. \begin{aligned} \ddot{x}_1 + x_1 + k(x_1 - x_2) + p_1 x_1^3 + q(x_1 - x_2)^3 &= 0 \\ \ddot{x}_2 + (1 + \alpha)x_2 + k(x_2 - x_1) + p_2 x_2^3 + q(x_2 - x_1)^3 &= 0 \end{aligned} \right\} \quad (1.1)$$

where α, k, p_1, p_2 and q are parameters, and $x_1, x_2 \in \mathcal{R}$. System (1.1) is called asymmetric one when $\alpha \neq 0$ or $p_1 \neq p_2$. When $\alpha = 0, p_1 = p_2$, system (1.1) is called symmetric one. Taking $p_2 = p_1 + \beta$, then we call α and β asymmetric parameters. We investigated nonlinear normal modes and their superposition of system (1.1) near the equilibrium point at origin.

II. Nonlinear Normal Modes and Their Singularity

An invariant set for a dynamical system is defined as a subset S of the phase space such that if the system is given an initial condition in S , the solution of the governing equations of motion remains in S all the time. According to the constructive method given by Shaw, we know they are invariant spaces for the nonlinear equation of motion and tangent to their linear counterparts, the planar eigenspace, at the equilibrium point. We rewrite system (1.1) as

$$\left. \begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= -x_1 - p_1 x_1^3 - k(x_1 - x_2) - q(x_1 - x_2)^3 \\ \dot{x}_2 &= y_2 \\ \dot{y}_2 &= -(1 + \alpha)x_2 - p_2 x_2^3 - k(x_2 - x_1) - q(x_2 - x_1)^3 \end{aligned} \right\} \quad (2.1)$$

or

$$\dot{z} = [A_0 + A_2(z)]z \quad (2.2)$$

where $z = [x_1, y_1, x_2, y_2]^T$,

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1+k) & 0 & k & 0 \\ 0 & 0 & 0 & 1 \\ k & 0 & -(1+\alpha+k) & 0 \end{bmatrix} \quad (2.3)$$

$$A_2(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -(p_1 + q)x_1^2 - 3qx_2^2 & 0 & 3qx_1^2 + qx_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ -3qx_2^2 + qx_1^2 & 0 & -(p_2 + q)x_2^2 - 3qx_1^2 & 0 \end{bmatrix}$$

a, k and q are not zero. $p_1 \neq p_2$.

Let $x_1 = u, y_1 = v$, and express x_2 and y_2 functionally in terms of u and v :

$$\left. \begin{aligned} x_2 &= a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2 + a_6 u^3 + a_7 u^2 v + a_8 uv^2 + a_9 v^3 + \dots \\ y_2 &= b_1 u + b_2 v + b_3 u^2 + b_4 uv + b_5 v^2 + b_6 u^3 + b_7 u^2 v + b_8 uv^2 + b_9 v^3 + \dots \end{aligned} \right\} \quad (2.4)$$

Substituting (2.4) into system (2.1) and gathering the coefficients of the same power of u and v yields a set of identical forms, where they are obtained by MATHEMATICA:

u term

$$\left. \begin{aligned} a_2 + b_1 + a_2 k - a_1 a_2 k &= 0 \\ -a_1 - \alpha a_1 + b_2 + k + b_2 k - a_1 k - a_1 b_2 k &= 0 \end{aligned} \right\} \quad (2.5)$$

v term

$$\left. \begin{aligned} -a_1 + b_2 - a_2^2 k &= 0 \\ -a_2 - \alpha a_2 - b_1 - a_2 k - a_2 b_2 k &= 0 \end{aligned} \right\} \quad (2.6)$$

u^2 term

$$\left. \begin{aligned} a_4 + b_3 - a_2 a_3 k + a_4 k - a_1 a_4 k &= 0 \\ -a_3 - \alpha a_3 + b_4 - a_3 k - a_3 k - a_3 b_2 k + b_4 k - a_1 b_4 k &= 0 \end{aligned} \right\} \quad (2.7)$$

uw term

$$\left. \begin{aligned} -2a_3 + 2a_5 + b_4 - 2a_2 a_4 k + 2a_5 k - 2a_1 a_5 k &= 0 \\ -a_4 - \alpha a_4 - 2b_3 + 2b_5 - a_4 k - a_4 b_2 k - a_2 b_4 k + 2b_5 k - 2a_1 b_5 k &= 0 \end{aligned} \right\} \quad (2.8)$$

v^2 term

$$\left. \begin{aligned} -a_4 + b_5 - 3a_2 a_5 k &= 0 \\ -a_5 - \alpha a_5 - b_4 - a_5 k - a_5 b_2 k - 2b_5 a_2 k &= 0 \end{aligned} \right\} \quad (2.9)$$

u^3 term

$$\left. \begin{aligned} a_7 + a_6 - a_3 a_4 k - a_2 a_6 k + a_7 k - a_1 a_7 k + a_2 p_1 + a_2 q \\ -3a_1 a_2 q + 3a_1^2 a_2 q - a_1^3 a_2 q &= 0 \\ -a_6 - \alpha a_6 + b_7 - a_6 k - a_6 b_2 k - a_3 b_4 k + b_7 k - a_1 b_7 k + b_2 p_1 - a_1^3 p_2 + q \\ -3a_1 q + 3a_1^2 q - a_1^3 q + b_2 q - 3a_1 b_2 q + 3a_1^2 b_2 q - a_1^3 b_2 q &= 0 \end{aligned} \right\} \quad (2.10)$$

$u^2 v$ term

$$\left. \begin{aligned} -3a_6 + 2a_8 + b_7 - a_4^2 k - 2a_3 a_5 k - 2a_2 a_7 k + 2a_8 k - 2a_1 a_8 k \\ -3a_2^2 q + 6a_1 a_2^2 q - 3a_1^2 a_2^2 q &= 0 \\ -a_7 - \alpha a_7 - 3b_6 + 2b_8 - a_7 k - a_7 b_2 k - a_4 b_4 k - 2a_3 b_5 k - a_2 b_7 k \\ + 2b_8 k - 2a_1 b_8 k - 3a_1^2 a_2 p_2 - 3a_2 q + 6a_1 a_2 q - 3a_1^2 a_2 q \\ + 3a_2 b_2 q + 6a_1 a_2 b_2 q - 3a_1^2 a_2 b_2 q &= 0 \end{aligned} \right\} \quad (2.11)$$

uw^2 term

$$\left. \begin{aligned} -2a_7 + 3a_9 + b_8 - 3a_4 a_5 k - 3a_2 a_8 k + 3a_9 k - 3a_1 a_9 k \\ + 3a_2^3 q - 3a_1 a_2^3 q &= 0 \\ -a_8 - \alpha a_8 - 2b_7 + 3b_9 + a_8 k - a_8 b_2 k - a_5 b_4 k - 2a_4 b_5 k - 2a_2 b_8 k \\ + 3b_9 k - 3a_1 b_9 k - 3a_1 a_2^2 p_2 + 3a_2^2 q - 3a_1 a_2^2 q \\ + 3a_2^2 b_2 q - 3a_1 a_2^2 b_2 q &= 0 \end{aligned} \right\} \quad (2.12)$$

v^3 term

$$\left. \begin{aligned} & -a_8 + b_9 - 2a_3^2k - 4a_2a_9k - a_2^4q = 0 \\ & -a_9 - \alpha a_9 - b_8 - a_9k - a_9b_2k - 2a_5b_5k - 3a_2b_9k - a_2^3p_2 \\ & -a_2^3q - a_2^3b_2q = 0 \end{aligned} \right\} \quad (2.13)$$

Noticing system (2.1) and expression (2.4), we may obtain $a_2 = b_1$. Substituting it into (2.5)~(2.13), all coefficients of two normal modes can be found by MATHEMATICA. If coefficients of model 1 is $a_{11}, a_{21}, a_{31}, \dots, b_{11}, b_{21}, b_{31}, \dots$, and ones of mode 2 is $a_{12}, a_{22}, a_{32}, \dots, b_{12}, b_{22}, b_{32}, \dots$ for system (2.1), then coefficients with cubic order approximation of two modes are respectively

Mode 1

$$\left. \begin{aligned} a_{11} &= b_{21} = -\alpha + \sqrt{\alpha^2 + 4k}/2k \\ a_{21} &= b_{11} = a_{31} = b_{31} = a_{41} = b_{41} = a_{51} = b_{51} = 0 \\ a_{61} &= \frac{-8k^4l_{61}n_{61}}{m_1^2}, a_{71} = 0, a_{81} = \frac{12g_{81}}{m_1} \\ a_{91} &= b_{61} = 0, b_{71} = \frac{3h_{71}j_{71}}{m_1}, b_{81} = 0, b_{91} = a_{81} \end{aligned} \right\} \quad (2.14)$$

Mode 2

$$\left. \begin{aligned} a_{12} &= b_{22} = \frac{-\alpha - \sqrt{\alpha^2 + 4k^2}}{2k} \\ a_{22} &= b_{12} = a_{32} = a_{42} = a_{52} = b_{32} = b_{42} = b_{52} = 0 \\ a_{62} &= \frac{-8k^4l_{62}n_{62}}{m_2^2}, a_{72} = 0, a_{82} = \frac{12g_{82}}{m_2} \\ a_{92} &= b_{62} = 0, b_{72} = \frac{3h_{72}j_{72}}{m_2}, b_{82} = 0, b_{92} = a_{82} \end{aligned} \right\} \quad (2.15)$$

where

$$\begin{aligned} l_{6i} &= (-62 - 3\alpha - 62k)\alpha^2 - (248 + 124\alpha - 248k)k^2 \\ &\quad \pm \sqrt{\alpha^2 + 4k^2}(48 + 48\alpha + 32\alpha^2 + 96k + 48\alpha k + 128k^2) \end{aligned} \quad (2.16)$$

$$\begin{aligned} n_{6i} &= \alpha k^3 p_1 - \alpha k(\alpha^2 + 3k^2)p_2 + (\alpha^4 + 2\alpha^3k + 4\alpha^2k^2 + 4\alpha k^3)q \\ &\quad \mp \sqrt{\alpha^2 + 4k^2}(-k^3p_1 + \alpha^2kp_2 + k^3p_2 - \alpha^3q - 2\alpha^2kq - 2\alpha k^2q) \end{aligned} \quad (2.17)$$

$$g_{8i} = n_{6i} \quad (2.18)$$

$$h_{7i} = 4 + 2\alpha + 4k \mp 4\sqrt{\alpha^2 + 4k^2} \quad (2.19)$$

$$j_{7i} = n_{6i} \quad (2.20)$$

$$m_i = -16k^4(1 + \alpha + 2k + \alpha k) + 4k^4(2 + \alpha + 2k \mp 2\sqrt{\alpha^2 + 4k^2}) \quad (2.21)$$

In (2.16)~(2.21), $i=1$ expresses coefficients of mode 1, $i=2$ ones of mode 2 and sign before $\sqrt{\alpha^2 + 4k^2}$ are taken correspondingly as over and below one. Thus, a three order approximate expression of nonlinear normal modes is

$$\left. \begin{aligned} x_2 &= a_{1i}u + a_{6i}u^3 + a_{8i}uv^2 \\ y_2 &= a_{1i}v + b_{7i}u^2v + a_{8i}v^3 \end{aligned} \right\} \quad (2.22)$$

where $i=1, 2$ are correspond to mode 1 and mode 2 respectively.

Fig. 1 shows the nonlinear invariant modal surfaces when $k = 1$, $\alpha = 0.05$, $p_1 = 0.5$, $p_2 = 0.7$, $q = 0.02$, where the thin surface is the first nonlinear normal mode and the thick surface the second one.

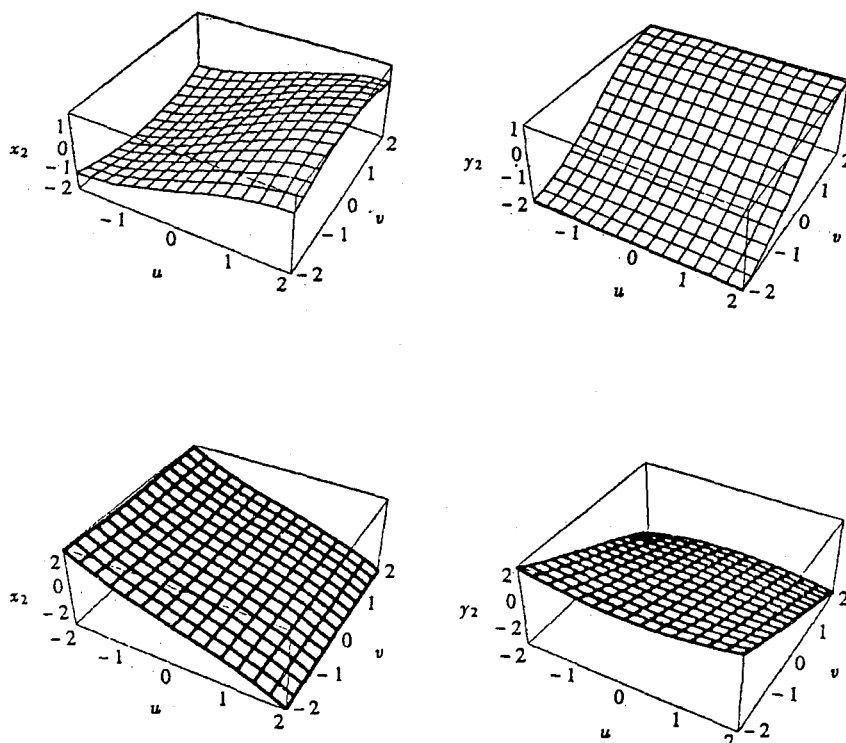


Fig. 1 The invariant surface of nonlinear normal modes (2.22)

Taking $\alpha = 0, p_1 = g, p_2 = 0, q = 0$ and substituting it into (2.14)~(2.21) yield coefficients of two modes.

Mode 1:

$$\left. \begin{aligned} a_{11} = b_{21} = 1, a_{21} = b_{11} = a_{31} = b_{31} = a_{41} = b_{41} = a_{51} = b_{51} = 0 \\ a_{61} = \frac{g(k-3)}{2k(k-4)}, a_{81} = -\frac{3g}{2k(k-4)}, b_{71} = \frac{3g(k-1)}{2k(k-4)} \\ b_{91} = a_{81}, a_{71} = b_{61} = b_{81} = a_{91} = 0 \end{aligned} \right\} \quad (2.23)$$

Mode 2:

$$\left. \begin{aligned} a_{12} = b_{22} = -1, a_{22} = b_{12} = a_{32} = b_{32} = a_{42} = b_{42} = a_{52} = b_{52} = 0 \\ a_{62} = \frac{(3+7k)g}{2k(4+9k)}, a_{82} = \frac{3g}{2k(4+9k)}, b_{72} = \frac{3g(1+3k)}{2k(4+9k)} \\ b_{92} = a_{82}, a_{72} = a_{92} = b_{62} = b_{82} = 0 \end{aligned} \right\} \quad (2.24)$$

Froms (2.23) and (2.24) have the same results as paper [6].

From expressions (2.14)~(2.21) of modal coefficients, it is seen that they appear to singularity when one of the following conditions is satisfied:

$$(1) k = 0 \quad (2.25)$$

$$(2) m_i = 0, \quad (i = 1, 2) \quad (2.26)$$

The above conditions (2.25) and (2.26) are called singularity ones of nonlinear normal modes for system (2.1). This paper only deals with nonsingularity for nonlinear normal modes. As for singularity, we will discuss it in the other paper.

III. Superposition in Nonlinear Normal Modes

It is well-known that the superposed solution of all modes for a linear system is also the solution of the system. They can be superposed. But superposition principle for a nonlinear system may be incorrect. The nonlinear normal modes defined here are in the neighborhood at the equilibrium point. How strong and weak nonlinearity is depends on how large or small the neighborhood is. The smaller neighborhood is, the weaker nonlinearity is. Therefore, nonlinear normal modes for superposition must be heavily dependent on the initial values. So, whether are the superposition of nonlinear normal modes affected by other factors? In the local view, one of the important distinctions between nonlinear and linear system is that there are bifurcations in nonlinear system^[11]. The discussed system (1.1) is a free vibration system and we consider that effects of static bifurcations of the equilibrium solution for the modal dynamical equation on degrees of nonlinear normal modes for superposition describing the original solution.

Letting (u_1, v_1) and (u_2, v_2) be the contribution of mode 1 and mode 2 to the displacement and velocity, then we can obtain a nonlinear relation between the physical coordinates and modal coordinates from (2.22). Substituting (2.22) into system (1.1) yields the modal dynamical equation that two modal coordinates present:

$$\ddot{u}_i + (1 + k - a_{1i}k)u_i + (q + p_1 - a_{6i}k - a_{1i}^3q + 3a_{1i}^2q - 3a_{1i}q)u_i^3 - a_{8i}ku_i\dot{u}_i^2 = 0 \quad (3.1)$$

where $i=1$ and $i=2$ correspond to mode 1 and mode 2 respectively. Eq. (3.1) shows that when

$$\frac{a_{1i}k - k - 1}{q + p_1 - a_{6i}k - a_{1i}^3q + 3a_{1i}^2q - 3a_{1i}q} > 0 \quad (3.2)$$

there are static bifurcations. Specially, when denominator of the left form in (3.2) is zero, degenerate bifurcations occur and more order approximation must be considered^[11]. Making data keep unity to be compared, we take $p_1 = 0.5$, $p_2 = 0.7$, $q = 0.02$ (see Fig. 1). Fig. 2 shows the static bifurcation of mode 1 and mode 2 following two parameters α and k for the modal dynamical Eq. (3.1), where the thick surface is the trivial solution and the thin surface nontrivial one.

If the solution of nonlinear normal mode 1 is noted as $x_2^{(1)}$, mode $x_2^{(2)}$ (see form (2.22)), then their superposed solution (noted as modal solution) is

$$\left. \begin{aligned} x_1 &= u_1 + u_2 \\ x_2 &= x_2^{(1)} + x_2^{(2)} \end{aligned} \right\} \quad (3.3)$$

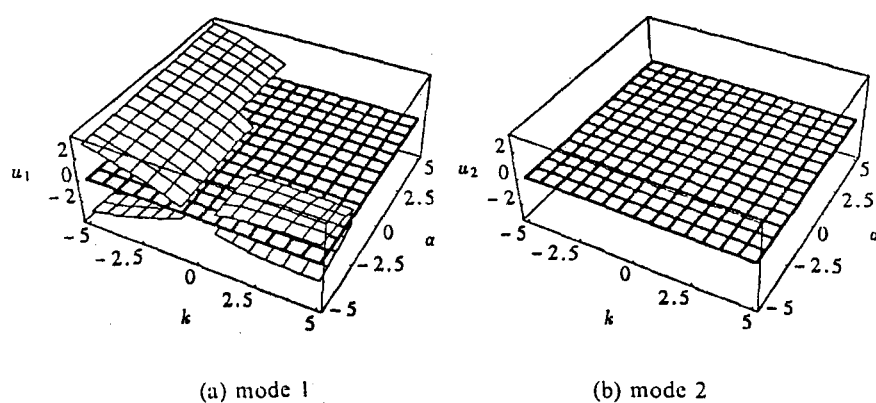


Fig. 2 Static bifurcation of equilibrium solution for modal dynamical equation following α, k

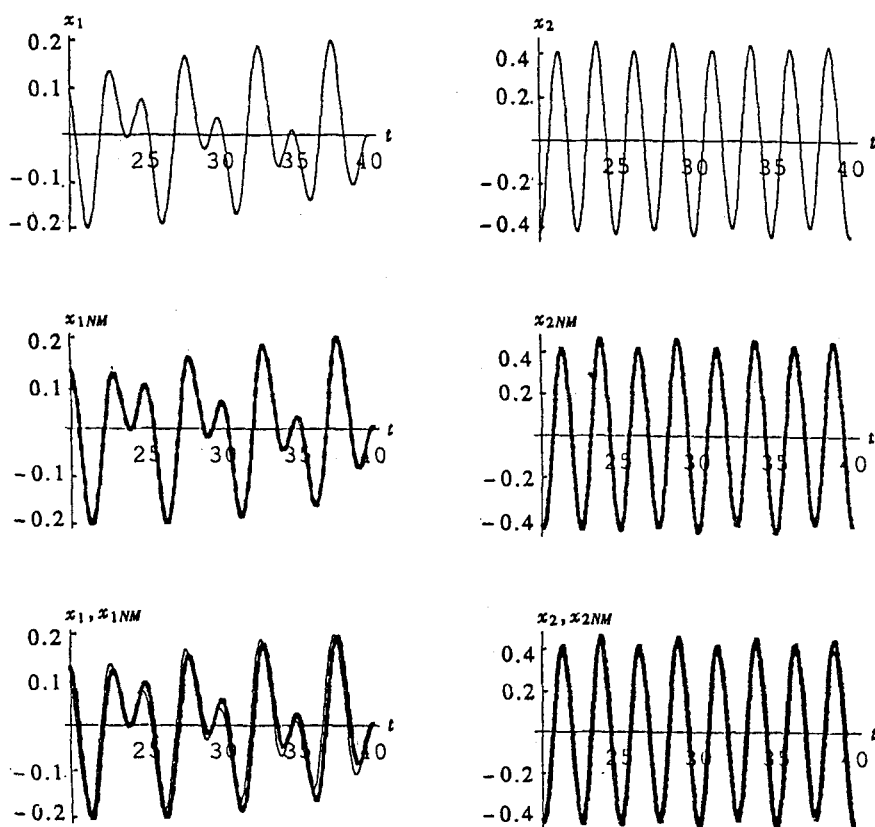


Fig. 3 Comparison between numerical and modal solution when $\alpha = 4, (u_{10}, v_{10}, u_{20}, v_{20}) = (0.1, 0, 0.1, 0)$

where u_1 and u_2 satisfy (3.1), and $x_2^{(1)}$ and $x_2^{(2)}$ are determined by (2.22). Next, we analyse degrees of the modal solution describing the original system (2.1). As for its accuracy, paper

[6] has had some good results, so we do not repeat here.

The analysis is done in three ways. First, system (2.1) is solved numerically in a four order RK method and the initial values are the same as those of modal solution (3.3). Then, the results of nonlinear normal modes are given. Finally, two results are compared. All results of the analysis are presented in Fig. 3, Fig. 4, Fig. 5 and Fig. 6. Fixing $k=1$ and taking p_1 and p_2 and q for the same value as those in Fig. 1 and Fig. 2. In Fig. 3 and Fig. 4, $\alpha=4$ and the initial values are respectively $(u_{10}, v_{10}, u_{20}, v_{20}) = (0.1, 0, 0.1, 0)$ and $(0.5, 0, 0.5, 0)$; In Fig. 5 and Fig. 6, $\alpha = -4$ and the initial values are respectively $(0.01, 0, 0.01, 0)$ and $(0.1, 0, 0.1, 0)$. Thus, the initial values of the corresponding modal solution and numerical solution may be obtained from (3.3). From Fig. 3 to Fig. 6, the thin line is the numerical solution, the thick line is the modal solution, and two traces are plotted together for comparison in the third row, where coefficients of the modes are given by using (2.14)~(2.21).

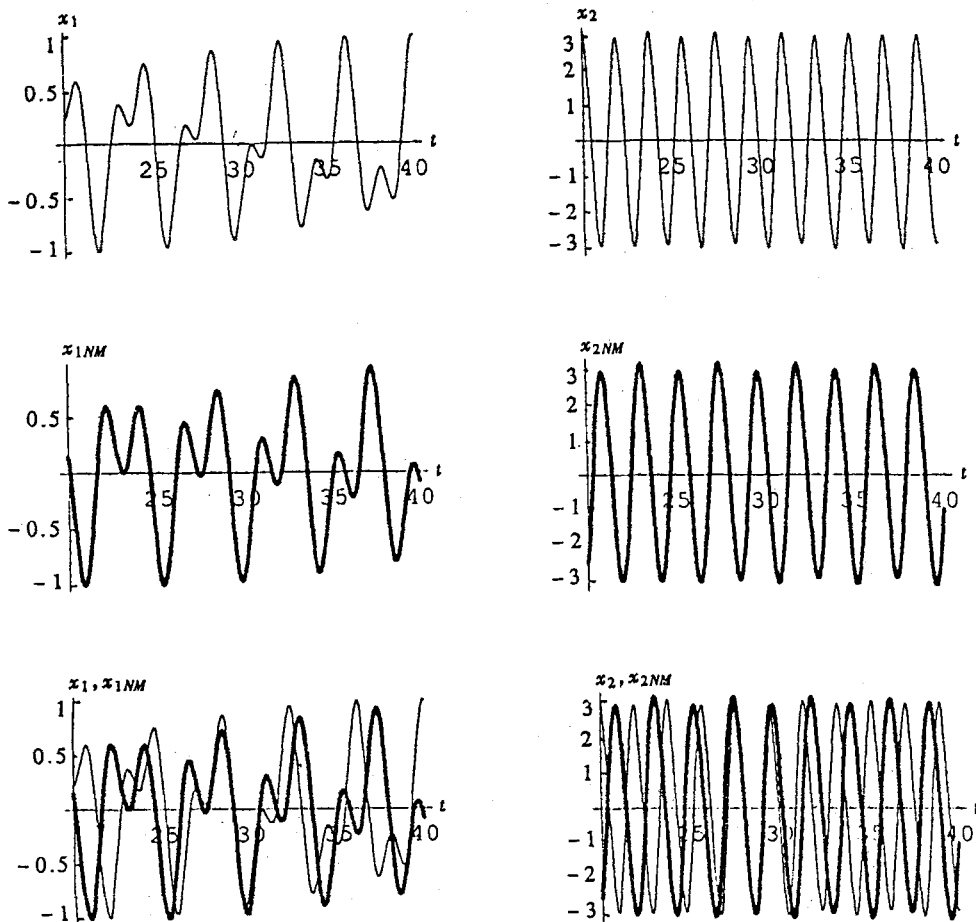


Fig. 4 .Comparison between numerical and modal solution when $\alpha = 4, (u_{10}, v_{10}, u_{20}, v_{20}) = (0.5, 0, 0.5, 0)$

Fig. 3 presents the numerical solution keeping a good unity with the modal solution and Fig. 4 shows that their unity is affected by the initial value increasing. However, the direction

of each curve is the same. Compared with numerical solution, the modal solution has a delay or lead at time. From Fig. 3 to Fig. 4, we may see that the method of nonlinear normal modes is suitable for a weak nonlinear system and it may be applied for engineering. But the method pends further study for a strong nonlinear system. As for comparison between modal superposed solution derived from nonlinear normal modes and corresponding linear solution, paper [6] has investigated. This paper does not repeat here. The unity is impossible in Fig. 5 and Fig. 6 although the chosen initial values are smaller at order of magnitude (see Fig. 2). Comparing them with Fig. 2(a), we may see that there is a static bifurcation of the equilibrium solution for the modal dynamical equation and it has an intrinsic effect on the modal solution. To examine view what we point out, we repeat the above analysis and investigate results at $\alpha = 0.05, \alpha = -0.05, \alpha = 2$, and $\alpha = -2$. In addition, Fixing $\alpha = 0.05$, we also study cases at $k=1, k=-1, k=-2, k=-4$. It is found out that unity between the numerical and modal solution is impossible how ever small the initial values are, when parameter α or k is chosen at those values where the static bifurcations occur. These present the modal solution is not able to describe the local dynamics of the original system. When α or k is chosen at those values where the bifurcations do not appear, unity between the numerical and modal solution is only affected by the initial values. The smaller initial values are, the better the unity is. These illustrate that the degrees of the modal solution

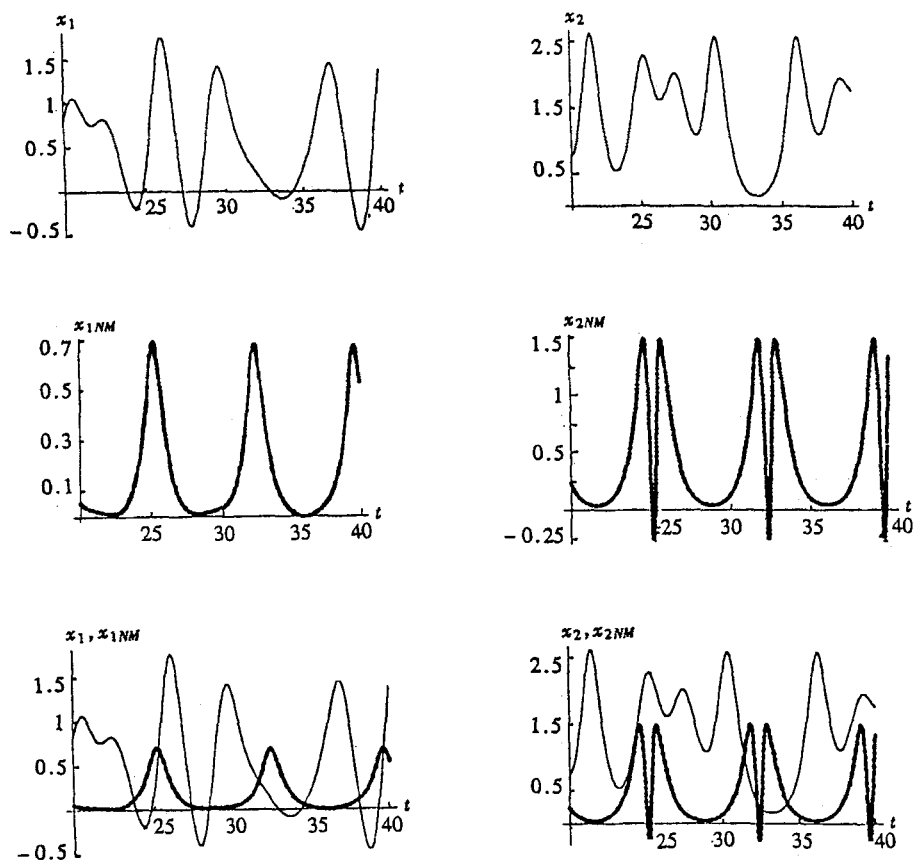


Fig. 5 Comparison between numerical and modal solution when $\alpha = -4, (u_{10}, v_{10}, u_{20}, v_{20}) = (0.01, 0, 0.01, 0)$

describing the original system only depend on the initial values. It also confirms that the method is valid in local space. The other figures have not appeared at this paper because pages are restricted.

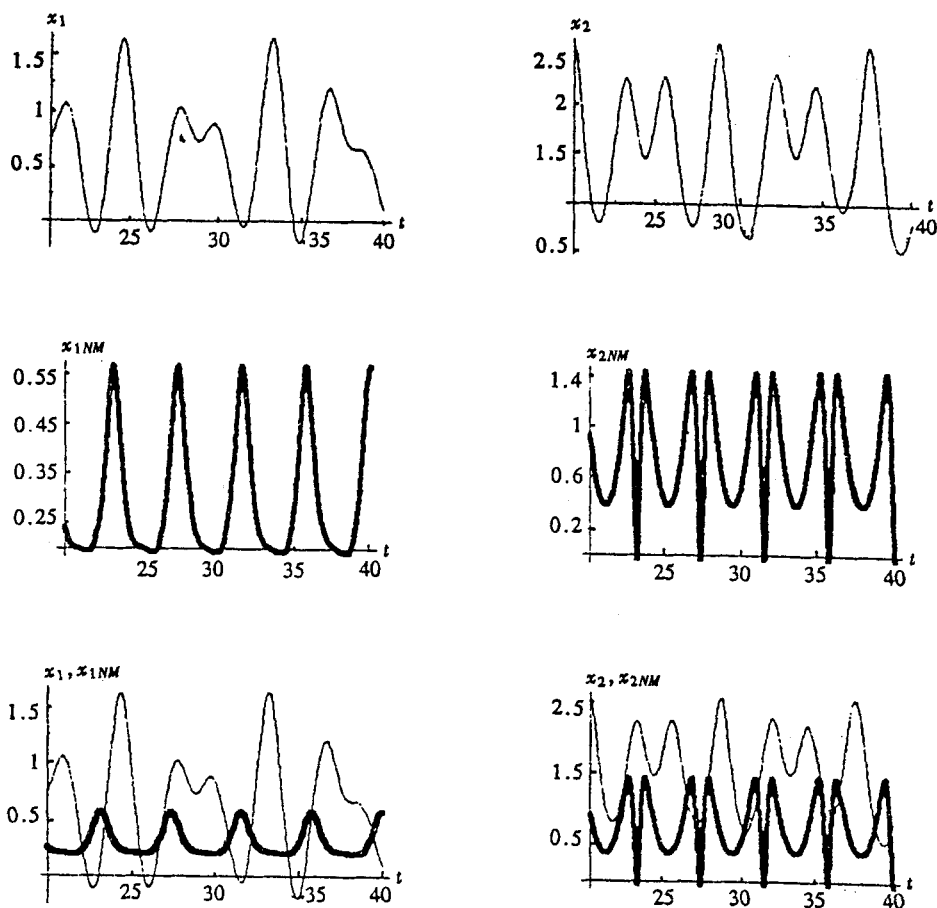


Fig. 6 Comparison between numerical and modal solution when $\alpha = -4, (u_{10}, v_{10}, u_{20}, v_{20}) = (0.1, 0, 0.1, 0)$

IV. Conclusions

(1) On nonsingularity, stability of equilibrium solutions for a modal dynamical equation that modal co-ordinates satisfy has close links with the degrees of nonlinear normal modes for superposition describing original systems.

(2) When parameters are chosen at those values where equilibrium solutions for a modal dynamical equation do not appear to static bifurcations, the degrees of modal solutions describing original systems depend heavily on initial values of the corresponding modal dynamical equation. The smaller initial values are, the better the degrees are. Conversely, when parameters are chosen at those values where equilibrium solutions for a modal dynamical equation appear to static bifurcations, the method of nonlinear normal modes is invalid.

(3) Properties of nonlinear normal modes for singularities pend further discussing.

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