Applied Mathematics and Mechanics
(English Edition, Vol.6, No.3, Mar. 1985)

Published by SUT,
Shanghai, China

THEOREM OF THE UNIQUENESS OF DISPLACEMENT AND STRESS FIELDS OF LINE-LOADED INTEGRAL EQUATION METHOD

Yun Tian-quan (云天铨)
(Huazhong University of Science and Technology, Wuhan)

(Received Apr. 12, 1984)

Abstract

According to Fredholm's theorem, this paper proves that due to the virtual fundaemental loads which satisfy the boundary conditions and being distributed outside the elastic body occupied region the displacement and stress fields in the elastic body occupied region are unique. This theorem forms a theoretical basis of the applications of the line-loaded integral equation method.

The line-loaded integral equation method is one of the integral equation methods in which the virtual fundamental loads are distributed in an elastic space (or half space) along a line segment outside the occupied region of an elastic solid and make the boundary conditions to be satisfied (the fundamental elastostatic equations are trivially satisfied), thus, the problem is reduced to an one-dimensional, non-singular integral equation. This method was used for many problems[1-5] with the advantage of simple calculation due to the one-dimensioness and non-singularity. However, the method itself is flexible for the virtual fundamental loads can be distributed in any region within the elastic space (or half-space) outside the solid. Consequently, a question arises are these displacement (or stress) fields due to two different distritions of a virtual load in which both cases the boundary conditions are satisfied the same? In the following, we answer this important question.

Let $\bar{R}$ be the closure of the occupied region of a solid in an elastic space. Let S be the boundary of R. For clearness, a displacement $\underline{u}_i$ is given over S. Suppose that there exists a virtual load with unknown intensity $x(Q)$ be so distributed on outside R that the boundary conditions are satisfied

$$u_i(P)=\int_\Omega U_i(P,Q)x(Q)dQ=\underline{u}_i \qquad (P\in S,\ Q\in\Omega) \tag{1}$$

are satisfied. Where $U_i(P, Q)$ is a known influence function which represents the displacement component ($i=1$, 2, 3.) at point P due to unit virtual fundamental load at point Q; $\bar{R}=R\cup S$, $\bar{R}\cap\Omega=\phi$. Then, the displacement and stress fields due to total virtual loads are:

$$u_i(P)=\int_\Omega U_i(P,Q)x(Q)dQ \qquad (P\in\bar{R},Q\in\Omega) \tag{2}$$

$$\sigma_{ij}(P)=\int_\Omega S_{ij}(P,Q)x(Q)dQ \qquad (P\in\bar{R},Q\in\Omega) \tag{3}$$

Where $S_{ij}(P,Q)$ is a stress influence function; $i, j=1$, 2, 3.

Remark: One of the characteristicSof our line-loaded integral equation method is that the virtual loads are distributed outside the elastic solid. That is, $Q\in\Omega$, $\Omega\cap\bar{R}=\phi$. According to this characteristic we prove the following lemma.

Lemma: If there exists a non-zero function $y(Q)$ orthogonal to a real non-symmetric kernel $A(P,Q)$ $(P\in S\subset\bar{R},\ Q\in\Omega,\ \bar{R}\cap\Omega=\phi)$, i.e.,

$$\int_{\Omega}A(P,Q)y(Q)dQ=0 \qquad (P\in S\subset\bar{R},\ Q\in\Omega,\ \bar{R}\cap\Omega=\phi) \tag{4}$$

then, $y(Q)$ orthogonal to the extension kernel $A(P,Q)$ $(P\in\bar{R},\ Q\in\Omega,\ \bar{R}\cap\Omega=\phi)$,i.e.,

$$\int_{\Omega}A(P,Q)y(Q)dQ=0 \qquad (P\in\bar{R},Q\in\Omega,\bar{R}\cap\Omega=\phi) \tag{5}$$

Proof:

In the classical theory of Fredholm's integral equation of the second kind, the domains of two variables of a kernel need to be the same. Now, the domain of the first variable P of kernel $A(P,Q)$ is $\bar{R}$, the domain of the second variable is Ω, and $\bar{R}\cap\Omega=\phi$. In order to make use of the Fredholm's theory, we rewrite the domains of the two variables to the same $M=\bar{R}\cup\Omega$, and (4):

$$\left.\begin{aligned}&\int_{M}A(u,v)y(v)dv=0 && (u,v\in M)\\ &\text{where}\quad A(u,v)=\begin{cases}A(P,Q), & \text{when}\ \ u\in S,\ v\in\Omega\\ 0, & \text{when}\ \ u\notin S\ \text{or}\ v\notin\Omega\end{cases}\end{aligned}\right\} \tag{6}$$

i.e., when any one of the variables does not lie in its domain, the kernel function does not exist (i.e., zero value). Similarly we can rewrite (5).

According to the following theorem[6]:

$$\int_{M}A(u,v)y(v)dv=0 \qquad (u,v\in M)\longleftrightarrow\langle y,\psi_n\rangle=0\ \ (\forall n) \tag{7}$$

i.e., the necessary and sufficient condition for a function y orthogonal to kernel A is y orthogonal to all characteristic functions of kernel A^*A. Where ψ_n satisfies the following homogeneous integral equation:

$$\psi_n(Q)=\lambda_n^2\int_M\int_M A^*(P,Q)A(P,t)\psi_n(t)dtdP \tag{8}$$

Where λ_n^2 is the characteristic value; $A^*(P,Q)=A(Q,P)$ is the adjoint kernel of A. Let

$$K(Q,t)=\int_M A(Q,P)A(P,t)dP \tag{9}$$

Then, (8) becomes:

$$\psi_n(Q)=\lambda_n^2\int_M K(Q,t)\psi_n(t)dt \tag{10}$$

Whatever $u\in\bar{R}$ or $u\in\Omega$, by (9), we have:

$$K(u,u)=\int_M A(u,P)A(P,u)dP=0 \tag{11}$$

Because that if $A(u, P)$ is defined for the first variable $u \in \bar{R}$, then $u \in \bar{R}$ is located outside the domain of the second variable of $A(P, u)$ and hence $A(P, u) = 0$; similarly, if $A(P, u)$ is defined for $u \in \Omega$, then $A(u, P) = 0$.

Similarly, we can prove that if $K(Q, t)$ of (9) exists, then $K(t, Q) = 0$; if $K(t, Q)$ exists, then $K(Q, t) = 0$. Because $A(P, t)$ of (9) is defined for $t \in \Omega$, then $A(t, P) = 0$ (for $t \in \Omega$ outside the domain of the first variable), thus

$$K(t,Q)=\int_M A(t,P)A(P,Q)dP \tag{12}$$

$K(t, Q) = 0$. Similarly, if $K(t, Q)$ exists, then $K(Q, t) = 0$. Consequently,

$$K(t,\ Q), K(t,Q)=0 \tag{13}$$

By (11), (13), the Fredholm determinant equals zero. i.e.,

$$K\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} \equiv \begin{vmatrix} K(s_1,s_1) & K(s_1,s_2) \\ K(s_2,s_1) & K(s_2,s_2) \end{vmatrix} = 0 \qquad (s_1, s_2 \in M) \tag{14}$$

Similarly, we have:

$$K\begin{pmatrix} s_1,\ s_2, \cdots, s_p \\ s_1,\ s_2, \cdots, s_p \end{pmatrix} = 0 \qquad (p=1,2,\cdots,s_p \in M) \tag{15}$$

Hence, the Fredholm's first series:

$$d(\lambda)=1+\sum_{p=1}^{\infty}\frac{(-\lambda)^p}{p!}\int_M \cdots \int_M K\begin{pmatrix} s_1,s_2,\cdots,s_p \\ s_1,s_2,\cdots,s_p \end{pmatrix} ds_1 ds_2 \cdots ds_p = 1 \neq 0 \tag{16}$$

According to Fredholm's first theorem[7], the homogeneous Fredholm integral equation (10) possesses only zero solutions in the domain M. i.e.,

$$\psi_n(Q)=0 \tag{17}$$

However, $\psi_n(Q)=0$ implies that

$$\psi_n(Q)=\lambda_n^2\int_M \int_M A^*(P,Q)A(P,t)\psi_n(t)dtdP \qquad (P \in \bar{R}) \tag{18}$$

what (18) holds is equivalent to:

$$\left.\begin{aligned} &\int_M A(u,v)y(v)dv=0 && (u,v \in M=\bar{R} \cup \Omega) \\ &A(u,v) = \begin{cases} A(P,Q) & (u \in \bar{R}, v \in \Omega) \\ 0 & (u \notin \bar{R} \text{ or } v \notin \Omega) \end{cases} \end{aligned}\right\} \tag{19}$$

(19) can be rewritten as:

$$\int_\Omega A(P,Q)y(Q)dQ=0 \qquad (P \in \bar{R}, Q \in \Omega, \bar{R} \cap \Omega = \phi) \tag{5}$$

(Q.E.D.)

Theorem: The displacement field (2) and stress field (3), due to virtual fundamental load distributions such that the boundary conditions (1) is satisfied, are unique.

Proof: Suppose that there are two different distributions of a virtual fundamental load with intensities x_1 and x_2 both such that the boundary conditions (1) is satisfied. The corresponding displacement and stress fields by (2) and (3), are denoted by u_{1i}, σ_{1ij} and u_{2i}, σ_{2ij} respectively.

Let $y=x_1-x_2$, by (2) and (3), we have:

$$\left.\begin{aligned}|u_{1i}(P)-u_{2i}(P)|&=\left|\int_{\Omega}U_i(P,Q)y(Q)dQ\right|\\|\sigma_{1ij}(P)-\sigma_{2ij}(P)|&=\left|\int_{\Omega}S_{ij}(P,Q)y(Q)dQ\right|\end{aligned}\right\}\quad(P\in\bar{R},Q\in\Omega)\qquad(20)$$

Because the boundary condition (1) is satisfied both by x_1 and x_2, we have:

$$\int_{\Omega}U_i(P,Q)y(Q)dQ=0\qquad(P\in S,Q\in\Omega)\qquad(21)$$

According to the lemma, we have:

$$\int_{\Omega}U_i(P,Q)y(Q)dQ=0\qquad(P\in\bar{R},Q\in\Omega)\qquad(22)$$

So that (20) gives:

$$u_{1i}(P)=u_{2i}(P)\qquad(P\in\bar{R})\qquad(23)$$

That is the displacement field is unique. According to the theory of elasticity, uniqueness of displacement field means the uniqueness of the stress field. (Q. E. D.)

If a stress boundary condition is given instead of the displacement boundary condition (1), we can prove the theorem by similar method.

References

[1] Yun Tian-quan, An integral equation mothod for solving the torsion problem of rotating bodies, *J. H. I. T.*, (English Edition), **1**, 1 (1979), 82 – 97. (MR 81 m: 73028).

[2] Yun Tian-quan, Pile analysis by simple integral equation method, *Applied Mathematics and Mechanics*, **2**, 3 (1981), 307 – 320, (MR 83i: 73014).

[3] Yun Tian-quan, Shao Uong-chang and Qiu Chong-guang, Analysis of elliposid compressed by two axial concentrated forces at two ends, *Applied Mathematics and Mechanics*, **2**, 6 (1981), 641 – 650.

[4] Yun Tian-quan, An integral equation method for problem of load acting perpendicularly to the surface of a half-space with rigid horizontal level *Acta Mechanica Solida Sinica*, 3 (1983), 375 – 383. (in Chinese)

[5] Yun Tian-quan, Integral equations its applications in elasticity, Lecture Notes of the 33rd symposium held under the auspices of Applied mathematics and mechanics, Shangdong Chemical Engineering Institute Press (1984).

[6] Smithies, F., *Integral Equations*, Cambridge University Press (1958).

[7] Pogorzelski, W., *Integral Equations and Their Applications*, Vol. 1, Pergamon Press, London (1966).