EXACT SOLUTION OF THE BOLTZMANN EQUATION

Shen Hui-chuan (沈惠川)

(*University of Science and Technology of China, Hefei*)

Abstract

We build up immediate connection between the nonlinear Boltzmann transport equation and the linear AKNS equation, and classify the Boltzmann equation as the Dirac equation by a new method for solving the Boltzmann equation out of keeping with the Chapman, Enskog and Grad's way in this paper. Without the effect of other external fields, the exact solution of the Boltzmann equation can be obtained by the inverse scattering method.

I. Introduction

.The AKNS equation[1-2] (i.e. Ablowitz-Kaup-Newell-Segur equation) is a basic equation in the theory of inverse scattering transformation[3-5]. The eigenvalue of AKNS equation in original sense is a dual spinor, and this AKNS equation is a dual Dirac equation[6-10]. This AKNS equation in original sense unites or induces some nonlinear dispersion problems in classical hydrodynamics, plasma dynamics, physics of elementary particles, nonlinear optics, physics of solid body, lattice mechanics and engineering. In ref. [11] we extend from this AKNS equation in original sense to the equation for four-spinor, and so as to solve the large deflection problem in nonlinear elastic mechanics. This AKNS equation for four-spinor is a four-Dirac equation in fact. We extend the application range for four-Dirac equation to the viscidity, the heat conduction and the diffusion of non-uniform gases, and build up immediate connection between the Boltzmann equation for non-uniform gases and the AKNS equation for four-spinor or the four Dirac equation, then turn the problem for exact solution of nonlinear Boltzmann equation into the problem for inverse scattering (general solution) of linear Dirac equation (or AKNS equation) in this paper.

L. Boltzmann is one of the "two great men that appeared in the previous century" by H. Haken[12] (another man is biologist C. Darwin). The nonlinear Boltzmann transport equation or the Boltzmann differential-integral equation (L. Boltzmann, 1872),[13] occupies a decisive position in hydrodynamics[14]. It can even provide the theoretic basic of hydrodynamics equations (such as the Navier-Stokes equation or its extension[15]. As early as 1879, J.C. Maxwell proved that[16] some other stresses exist in very rarefied static gases with non-uniform temperature and are not calculated in the Navier-stokes equation. From this, in many classical books of hydrodynamics[14], the authors must tell the Boltzmann equation and its solution before the text of the books. The Boltzmann equation also has the first important role in the molecular kinetic theory of gases[17-22] classical non-equilibrium statistical thermodynamics[23-24], plasma physics and physics of solid body[25]. To solve the Boltzmann equation is at all times a mechanical theorizer's own job.

The basic thinking for the Boltzmann equation was put forward first by Maxwell (1866)[16], and

was formulated by Boltzmann[26-27]. For the sake of emphasis on the first work by Maxwell, this equation is called the Maxwell-Boltzmann equation by D. Hillbert[28]. The Boltzmann equation is fairly exact in application to rarefied gases[23]. This explains that the Boltzmann equation mirrors the objective truth in some degree. But the application range for the Boltzmann equation is only in this condition. Up to now, any non-suitable extension to the Boltzmann equation has ended in failure[23]. In addition, on standard standpoint of statistical mechanics, by the Boltzmann equation we cannot describe "the fluctuations"[20] and the Boltzmann equation is excluded from classical theory of statistical mechanics in fact. In modern statistical mechanics, the leading equation is the Liouville equation[17-24, 29] and the Boltzmann equation only plays a small inconspicuous part in it. This unjust attitude to the Boltzmann equation makes some indignant authors[16] firmly support the Boltzmann equation with the aim of "uprising", and of independence of the kingdom of statistical mechanics to a style of its own, and think that "this need not be explained any more".

The first solution of the Boltzmann equation was obtained independently by S. Chapman and by D. Enskog[16] in about a year (1916-1917). The method[20] of Chapman and Enskog does not immediately solve the Boltzmann equation, but looks upon the probability-distributed function as the function of observable macroscopic number density of moleucls, observable macroscopic mean velocity and observable macroscopic temperature; the observable macroscopic number density of molecules, the observable macroscopic mean velocity and the observable macroscopic temperature are obtained from the hydrodynamics equations. The hydrodynamics equations as the differential equations in three-dimensional coordinate space, by contrast to the Boltzmann differential-integral equation in six-dimensional μ-space, are simple. In it, the theoretical system for Chapman slants to direct perception through the senses, and does not lay stress on the systematicness and the deduction; oppositely, the discussion for Enskog lays more stress on the form and the style of mathematics. Their methods are irrelevalnt on the thinking or on the details, but their results are absolutely identical.

An up-and-coming youngster H. Grad (1949)[30] discovered a new method for solving the Boltzmann equation. His result can be applied to condition such as within the shock wave, i.e. to condition of prompt change for probability-distributed function along with space and time. In the theory of Grad the thermal flux – vector and the stress tensor find themselves in an equal position with other unknown quantity, such as pressure, density, velocity and so on. The method of Grad approaches the method of Enskog, and the principal difference between them rests with the discussion on time-derivative. The solution of Grad has still more general form over the solution of Enskog. Tsien[14] Hsue-shen thinks that, the theory of Grad has special important application in dynamics of rarefied gases.

In addition, Grad[31-32], Kihara[33], Waldmann[34], Balescu[35], Hirschfelder-Curtiss-Bird[36] and Cercignani[37-40] et al[4-43], also make many works on solving the Boltzmann equation. But these papers and monographs are not so good as refs.[16, 30] in reputation, because the methods are outmoded.

All the above-mentioned solution methods are the approximate methods for solving the Boltzmann equation. Owing to difficulty in mathematics, as early as the preceding century, L. Boltzmann explained that[16] people should abandon the hopes on the general solution for solving this equation.

Now we are aware that by available strength the general solution of only a few equations in nonlinear equations (such as the Burgers equation and its extension[4]) can be obtained from mathematical transformation. For the nonlinear equations of large number we can only obtain their

exact special solution up to now. The Boltzmann equation is a nonlinear differential-integral equation; we ponder a trifle, and can find that it is also rather ordered. For this reason, we can completely obtain its exact solution from advanced mathematical method. This advanced mathematical method is the theory of inverse scattering transformation at the beginning of the paper.

For the sake of solving the exact solution of the Boltzmann equation by the theory of inverse scattering transformation, we must first build up connection between the nonlinear Boltzmann equation and the linear Dirac equation. The AKNS equation for four-spinor, just as is said, is the four-Dirac equation. The major work and objective in this paper lies in building up connection between them.

The Dirac equation is an important equation with extensive applicable range, and it is one basic equation in nature. In this paper we bring the nonlinear Boltzmann equation into the category of Dirac equation, thus not only for the sake of making the solution of nonlinear problems easy, but also for enabling nature to become even more harmonious and synergetic[44], and furthermore for enabling the language of describing the natural phenomenon to become even simpler and more normalizable. For this reason, the method in this paper has sense not only in methodology, but also in philosophy.

Before solving the Boltzmann equation by the theory of inverse scattering transformation, we also pay attention to prerequisite condition for application of this theory, that the probability-distributed function must suddenly drop in distant place or satisfy the periodic boundary condition[3-5]. The former is very obvious in our problem, and can after all be satisfied. (Incidentally, in solving the von Karman equation for nonlinear elastic large deflection problem by the theory of inverse scattering transformation[11,45-47], we also have the same problem. At this time the prerequisite condition for application of this theory can also be satisfied, or try to be satisfied.

It must be pointed out that the obtainable solution from solving the Boltzmann equation by the theory of inverse scattering transformation is the exact solution and is not the general solution, then in building up connection between the nonlinear Boltzmann equation and the linear Dirac equation (AKNS equation) we need not claim that every step of the advance must satisfy the ample-essential condition, but only claim that the later obtainable equation is ample condition for the above equation. Naturally, the exact solutions have difference on the "exact" degree. For this reason people must seize the sense of propriety for ample condition in deduction process. For the sake of improving on exactitude of solution we must omit as few factors as possible and must slow down the full substitution of the initial-boundary conditions into the equations. But, for convenience sake we still omit some factors. In this paper for convenience sake we apply the so-called "Maxwell hypothesis", i.e. the scattering cross-section has nothing to do with the relative momentum[20]. On this hypothesis, some of the secondary factors are omitted. In calculation for the probability-distributed function with the molecular action force depend directly on r^5 (where r is the distance between two molecules), Maxwell applied this hypothesis[48-50]. The Maxwell hypothesis is a still simpler additional condition of the Boltzmann equation in fact. If there is not one additional condition or another, then we are unable to solve the Boltzmann equation.

The similar method of inverse scattering transformation can be applied to solve the Vlasov equation[20] in plasma physics. This Vlasov equation is a special case of BBGKY equation (Bogolinbov[50]-Born-Green[51], -Kirkwood[50,52]-Yvon[54] equation) for the theory of dense gases in fact. The discussion of this problem is largely identical but with minor differences in this paper. It is unnecessary to go into details.

The dummy index is the summation depending on the Einstein convention in this paper.

II. Relation Between the Boltzmann Equation and the Nonlinear Schrödinger Equation

The physical event that is described by the Boltzmann equation finds itself in six-dimensional μ - space. The general six-dimensional μ -space is a direct sum for the three-dimensional coordinate space and the three-dimensional velocity space[20]. But more convenient is the six-dimensional canonical μ - space, which is a direct sum for three-dimensional coordinate space and the three-dimensional momentum space. From ref. [19] for the Boltzmann equation we write

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{\text{Coll.}} \tag{2.1}$$

where

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x}^k \frac{\partial f}{\partial x^k} + \dot{p}^k \frac{\partial f}{\partial p^k}$$

$$= \frac{\partial f}{\partial t} + \frac{1}{m} p^k \frac{\partial f}{\partial x^k} + F^k \frac{\partial f}{\partial p^k} \qquad (k=1, 2, 3) \tag{2.2}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{Coll}} = \iint \sigma d\Omega (f'f_1' - ff_1)|g| d^3 p_1 \tag{2.3}$$

g is the relative momentum

$$g = p - p_1 \quad , \quad |g| = \sqrt{(p^k - p_1^k)^2} \tag{2.4}$$

and $\sigma = \sigma(|g|, \theta)$ is the scattering cross-section, θ is the angle of refraction for g, $d\Omega$ is the unit of solid angle, m is the molecular mass, p^k is the component of momentum, F^k is the component of external field force. That the external field force has incorporated not only the binding force of external world, but also the molecular mutual action force[16]. At this time f is a probaility-distributed function, and

$$\left.\begin{array}{ll} f=f(p^k) & , \quad f'=f(P^k) \\ f_1=f(p_1^k) & , \quad f_1'=f(P_1^k) \end{array}\right\} \tag{2.5}$$

where p^k and p_1^k are the components of momentum prior to scattering, and P^k and P_1^k are components of momentum after scattering ($k=1,2,3$).

The Boltzmann equation (2.1) is the irreversible equation. In operation on $t \to - t$, we have $p^k \to - p^k, F^k \to F^k$. At this time the sign of the left side of equal-sign for equation (2.1) changes, and the sign of the right side does not change.

The original expression of the scattering term (2.3) form Jacobian $\dfrac{\partial(P^k, P_1^k)}{\partial(p^k, p_1^k)} = 1$ can be written as[20]

$$\left(\frac{\partial f}{\partial t}\right)_{\text{Coll.}} = f' \int \sigma d\Omega \int f_1' |g| d^3 p_1 - f \int \sigma d\Omega \int f_1 |g| d^3 p_1 \tag{2.6}$$

i.e.

$$\left(\frac{\partial f}{\partial t}\right)_{\text{Coll.}} = \text{Re}[-(f-if')\int \sigma d\Omega \int (f_1 - if_1')|g| d^3 p_1] \tag{2.7}$$

where Re is expressed as the real part of formula.

We let

$$\psi = \frac{1}{2}(f - if') \quad , \quad \psi_1 = \frac{1}{2}(f_1 - if_1') \tag{2.8}$$

where the independent variables in f, f_1, f', f_1' except for t, x^k are also p^k, p_1^k, P^k, P_1^k respectively and the independent variables in ψ, ψ_1 except for t, x^k are also p^k, p_1^k respectively. The transformation relation between independent variables in (2.8) follows the old routine of original formula for the Boltzmann equation. At this time, equation (2.1) can be written as

$$\mathrm{Re}\ \frac{d\psi}{dt} = 2\mathrm{Re}[-\psi \int \sigma d\Omega \int \psi_1 |g| d^3 p_1] \tag{2.9}$$

The sufficient condition (we can prove that it is the sufficiently essential condition) for equation (2.9) to be valid is

$$\frac{d\psi}{dt} = -2\psi \int \sigma d\Omega \int \psi_1 |g| d^3 p_1 \tag{2.10}$$

i.e.

$$\frac{\partial \psi}{\partial t} + \dot{x}^k \frac{\partial \psi}{\partial x^k} + \dot{p}^k \frac{2\psi}{\partial p^k} = -2\psi \int \sigma d\Omega \int \psi_1 |g| d^3 p_1 \quad (k=1, 2, 3) \tag{2.11}$$

or

$$\frac{\partial \psi}{\partial t} + \frac{1}{m} p^k \frac{\partial \psi}{\partial x^k} + F^k \frac{\partial \psi}{\partial p^k} = -2\psi \int \sigma d\Omega \int \psi_1 |g| d^3 p_1 \quad (k=1, 2, 3) \tag{2.12}$$

We introduce the integral operator $\hat{\jmath}$

$$\hat{\jmath} = -i \int \sigma d\Omega \int d^3 p_1 |g| \tag{2.13}$$

Look out, the integral operator $\hat{\jmath}$ is complex. From (2.13) we can write the Boltzmann equation (2.11) or (2.12) in simplified form

$$\frac{\partial \psi}{\partial t} + \dot{x}^k \frac{\partial \psi}{\partial x^k} + \dot{p}^k \frac{\partial \psi}{\partial p^k} = -2i\psi(\hat{\jmath}\psi) \tag{2.14}$$

and

$$\frac{\partial \psi}{\partial t} + \frac{1}{m} p^k \frac{\partial \psi}{\partial x^k} + F^k \frac{\partial \psi}{\partial p^k} = -2i\psi(\hat{\jmath}\psi) \tag{2.15}$$

where

$$\hat{\jmath}\psi = -i \int \sigma d\Omega \int \psi_1 |g| d^3 p_1 \tag{2.16}$$

In addition, we record

$$\hat{L} = -i\dot{x}^k \frac{\partial}{\partial x^k} - i\dot{p}^k \frac{\partial}{\partial p^k} + 2\nu \tag{2.17}$$

$$\nu = \hat{\jmath}\psi$$

then for the Boltzmann equation (2.14) we can write the form of the Schrödinger equation

$$i\,\frac{\partial\psi}{\partial t}=\hat{L}\psi \tag{2.18}$$

We can prove that not only equation (2.18) has form of the Schrödinger equation, but also its operator $\hat{L}$ is Hermitian operator as the energy operator in the Schröddinger equation. In this connection, we apply the symbol

$$l_{mn}=\int\bar{G}_m(\hat{L}G_n)dpdx \tag{2.19}$$

where G_m and G_n are the arbitrary functions of p and x, and they are integrable on the phase space; $\bar{G}_m$ is complex conjugate of G_m. And the condition for hermiticity is

$$l_{mn}=\bar{l}_{nm} \tag{2.20}$$

(2.20) means all the diagonal elements l_{mn} are real. This test is straightforward. From integration by parts, and associated with G_m and G_n equal to zero on bound, then we can obtain

$$l_{mn}=\int\bar{G}_m(\hat{L}G_n)dpdx$$

$$=i\int\bar{G}_m\left[-\dot{x}^k\frac{\partial}{\partial x^k}-\dot{p}^k\frac{\partial}{\partial p^k}-2i\nu\right]G_ndpdx$$

$$=-i\int G_n\left[-\dot{x}^k\frac{\partial}{\partial x^k}-\dot{p}^k\frac{\partial}{\partial p^k}-2i\nu\right]\bar{G}_mdpdx=\bar{l}_{nm} \tag{2.21}$$

Thus, the Boltzmann equation (2.14) or (2.15) can become the standard nonlinear Schrödinger equation. But look out, the normalizing condition of equation (2.18) is different with the normalizing condition of Schrödinger equation in quantum mechanics. In the Boltzmann equation,

$$\int fd^3p=n(x^k,\ t) \tag{2.22}$$

$$\int n(x^k,\ t)d^3x=N \tag{2.23}$$

where n is the number density of molecules in time t and corrdinate $x^k(k=1,2,3)$ and N is total number of molecules of gases. Then, the normalizing condition of equation (2.18) becomes

$$\int(\psi+\bar{\psi})d^3pd\ x=N \tag{2.24}$$

Because the scattering potential is

$$\nu=\hat{J}\psi \tag{2.25}$$

Thus equation (2.18) has the form of the nonlinear Schrödinger equation.

III. Scattering Potential for the Boltzmann Equation

Owing to the Maxwell hypothesis, from the scattering potential for the Boltzmann equation (2.18) i.e.

$$\nu=\hat{J}\psi \tag{3.1}$$

we can obtain the equation

$$\nabla_{p}^{4}\nu = A\psi \qquad (3.2)$$

where A is an arbitrary constant, and

$$\nabla_{p} = \left(\frac{\partial}{\partial p^1}, \ \frac{\partial}{\partial p^2}, \ \frac{\partial}{\partial p^3}\right) \qquad (3.3)$$

Equation (3.2) can be obtained by the classical equations for mathematical physics[55]. In the calssical equations for mathematical physics, from the n-dimensional m-layered harmonic equation

$$\Delta^m u = 0 \qquad (3.4)$$

we can obtain its basic solution as

$$\nu(x^k, \ x_1^k) = \begin{cases} cr^{2m-n}\ln r & (2m \geqslant n, \ n \text{ is even number}) \\ cr^{2m-n} & (\text{other condition}) \end{cases} \qquad (3.5)$$

where

$$r = \sqrt{(x^k - x_1^k)^2} \qquad (3.6)$$

Then, from the n-dimensional m-layered Poisson equation

$$\Delta^m u = w \qquad (3.7)$$

we can obtain its special solution on second condition in (3.5) as

$$u = c \int w(x_1^k) r^{2m-n} d^3 x_1 \qquad (3.8)$$

We contrast equation (3.8) with equation (3.1). Now in our problem,

$$n = 3 \ , \quad m = 2 \qquad (3.9)$$

and

$$x^k \to p^k \qquad (k=1, \ 2, \ 3) \qquad (3.10)$$

Thus, we can directly obtain (3.2) from (3.7).

VI. Schrödinger's Representation for the Boltzmann Equation

As the Boltzmann equation (2.1) can be written as the form of the Schrödinger equation (2.18) by transformation (2.8), and turned into the eigenvalues problem, the eigenfunction in the Schrödinger's representation can generally be written as

$$\psi = \exp[i(p^k x^k - Et)] \qquad (k=1, \ 2, \ 3) \qquad (4.1)$$

If the operator $\left(-i\dfrac{\partial}{\partial x^k}\right)$ acts on the left side and the right side of equal-sign for (4.1) at the same time, we have

$$-i\frac{\partial}{\partial x^k}\psi = p^k\psi$$

Thus in the Schrödinger's coordinate picture we have

$$\hat{p}^k = -i\frac{\partial}{\partial x^k} \qquad (4.2a)$$

In addition, from the momentum picture in quantum mechanics[56] we can know

$$\hat{x}^k = i\frac{\partial}{\partial p^k} \tag{4.2b}$$

Now we backwardly study equation (2.18) again. We notice that the independent variables of equation (2.18) except for time t are conjugate mechanical quantity x^k and p^k ($k = 1,2,3$). Secondly, as the f or ψ is the expression of the probability-distributed function and can be applied to calculation for other mechanical quantity, we can regard them as the operators. From the two above-mentioned grounds we can think that, although equation (2.18) has the form of the Schrödinger equation, it can be regarded as the operator equation in the Heisenberg representation at the same time.

To turn the operator equation in the Heisenberg representation in to the eigen-equation in the Schrödinger's representation, we can apply equality (4.2). For the sake of caution, we apply the ready-made method[56] in deduction process.

We multiply equation (2.12) at the same time · by the statevector: $(bra)\langle a_h|$ and $(ket)|\beta_h\rangle$, and write ψ of equation (2.18) as ψ^h, write ψ_1 as ψ_1^h where index h is the expression for the Heisenberg representation, and $\langle|$ or $|\rangle$ is the Dirac symbol[7]. At this time we have the equation for the matrix element:

$$\langle a_h|i\frac{\partial\psi^h}{\partial t}|\beta_h\rangle + \langle a_h|\left(i\frac{1}{m}p^k\frac{\partial}{\partial x^k} + iF^k\frac{\partial}{\partial p^k}\right)\psi^h|\beta_h\rangle$$
$$= -2i\langle a_h|\psi^h\int\sigma d\Omega\int\psi_1^h|g|d^3p_1|\beta_h\rangle \tag{4.3}$$

Let

$$\psi^h = \exp[iHt]\psi^s\exp[-iHt] \tag{4.4}$$
$$|a_h\rangle = |a_s(0)\rangle = \exp[iHt]|a_s(t)\rangle \tag{4.5}$$

where H is the total Hamiltonian, and index s is the expression for the Schrödinger's representation; H does not contain clear time t, and it is constant on definite ψ

Substitute (4.4) and (4.5) into equation (4.3), and the right side of equal-sign in it becomes

$$-2i\langle a_h|\psi^h\int\sigma d\Omega\int\psi_1^h|g|d^3p_1|\beta_h\rangle$$
$$= -2i\langle a_s(0)|\exp[iHt]\psi^s\exp[-iHt]\int\sigma d\Omega$$
$$\cdot\int\exp[iH_1t]\psi_1^s\exp[-iH_1t]|g|d^3p_1|\beta_s(0)\rangle \tag{4.6}$$

where $p_1^k(k=1, 2, 3)$ only have the meaning of parameter at this time, and

$$H = H(x^k, -i\partial^k)$$
$$H_1 = H(x_1^k, -i\partial_1^k) \tag{4.7}$$

Owing to the law of conservation of energy, we have

$$H_1 = H \tag{4.8}$$

then

$$-2i\langle a_h|\psi^h\int\sigma d\Omega\int\psi_1^h|g|d^3p_1|\beta_h\rangle$$

$$= -2i\langle a_s(0)| \exp[iHt]\psi^s\exp[-iHt]\int \sigma d\Omega$$

$$\cdot \int \exp[iHt]\psi_1^s\exp[-iHt]|g|d^3p_1|\beta_s(0)\rangle$$

$$= -2i\langle a_s(0)|(\exp[iHt]\psi^s\exp[-iHt])(\exp[iHt]\int \sigma d\Omega$$

$$\cdot \int \psi_1^s|g|d^3p_1\exp[-iHt])|\beta_s(0)\rangle$$

$$= -2i\langle a_s(0)|\exp[iHt][\psi^s\int \sigma d\Omega\int \psi_1^s|g|d^3p_1]\exp[-iHt]|\beta_s(0)\rangle \qquad (4.9)$$

Thus (4.3) becomes

$$\langle a_s(0)|\exp[iHt]i\frac{\partial\psi^s}{\partial t}\exp[-iHt]|\beta_s(0)\rangle$$

$$+\langle a_s(0)|\exp[iHt]\left(i\frac{1}{m}p^k\frac{\partial}{\partial x^k}+iF^k\frac{\partial}{\partial p^k}\right)\psi^s\exp[-iHt]|\beta_s(0)\rangle$$

$$= -2i\langle a_s(0)|\exp[iHt][\psi^s\int \sigma d\Omega\int \psi_1^s|g|d^3p_1]\exp[-iHt]|\beta_s(0)\rangle \qquad (4.10)$$

From

$$\left.\begin{array}{l} |a_s(t)\rangle = \exp[-iHt]|a_s(0)\rangle \\ \langle a_s(t)| = \langle a_s(0)|\exp[iHt] \end{array}\right\} \qquad (4.11)$$

and attend to (4.2), then for (4.10) we can write

$$\langle a_s| i\frac{\partial\psi^s}{\partial t} |\beta_s\rangle + \langle a_s|\left(\frac{1}{m}\nabla^2-2V\right)\psi^s|\beta_s\rangle$$

$$= -2i\langle a_s| \psi^s\int \sigma d\Omega\int \psi_1^s|g|d^3p_1|\beta_s\rangle \qquad (4.12)$$

where V is the Clausius Virial[57-58].

$$V = -\frac{1}{2}F^kx^k \qquad (k=1, 2, 3) \qquad (4.13)$$

This is different from quantum mechanics.

As equality (4.12) is always tenable on all $\langle a_s|$ and $|\beta_s\rangle$, we have

$$i\frac{\partial\psi^s}{\partial t} = \left(-\frac{1}{m}\nabla^2+2V\right)\psi^s - 2i\psi^s\int \sigma d\Omega\int \psi_1^s|g|d^3p_1 \qquad (4.14)$$

Omit index s, and follow the example of the integral operator (2.13); in integral operator $p_1^k(k=1, 2, 3)$ are the parameters only, thus we have

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{m}\nabla^2+2V\right)\psi + 2\psi(\hat{J}\psi) \qquad (4.15)$$

By now, we have already turned the Boltzmann equation (2.1) into the nonlinear Schrödinger equation. All the processes of deduction are sufficiently essential. Our obtainable nonlinear Schrödinger equation (4.15) is different from the nonlinear Schrödinger equation in quantum mechanics only in the following two aspects: the first is that the normalizing condition only satisfies equation (2.24) but does not satisfy $\int \bar{\psi}\psi d\tau = 1$; the second is the appearance of the Clausius Virial

of external field but not the appearance of the potential function of external field, i.e. the relation between energy and momentum for gaseous molecule becomes complex.

In addition, we must notice that equation (4.15) is still irreversible as a result of the complex operator $\hat{J}$. Equation (4.15) corresponds to the Schrödinger equation with complex energy[56] This Schrödinger equation with complex energy is also an irreversible equation.

In the following, from this we solve the exact solution of the Boltzmann equation. For convenience sake, temporarily we hypothesize that the received external force on molecule is zero, and Virial $V=0$

V. Exact Solution of the Boltzmann Equation

Let the molecular mass $m=1$, and temporarily omit the Virial of external field. At this time the Boltzmann equation (2.1) equal

$$i\frac{\partial \psi}{\partial t}+\nabla^2\psi-2\psi(J\psi)=0 \qquad (5.1)$$

where we omit symbol $\wedge$ on operator $\hat{J}$

We act on the left side and the right side of equal-sign for equation(5.1) by operator J at the same time, and have

$$\left[i\frac{\partial}{\partial t}J+\nabla^2J-2(J\psi)J\right]\psi=0 \qquad (5.2)$$

look out, for the second action of operator J on ψ , $|g|=\sqrt{(p_2^1-p^k)^2}$

We take the complex conjugate for (5.2), and have

$$\left[i\frac{\partial}{\partial t}\bar{J}-\nabla^2\bar{J}+2(\bar{J}\bar{\psi})\bar{J}\right]\bar{\psi}=0 \qquad (5.3)$$

Now we can write equation (5.1) and (5.3) as follows

$$i\frac{\partial q}{\partial t}+\nabla^2q-2qrq=0 \qquad (5.4a)$$

$$i\frac{\partial \bar{r}}{\partial t}-\nabla^2\bar{r}+2\bar{r}\bar{q}\bar{r}=0 \qquad (5.4b)$$

where

$$q=\psi \quad ; \quad r=J \qquad (5.5)$$

In equations (5.4a) and (5.4b) we can find the following rule: we exchange q and r in equation (5.4b) and take the complex conjugate for the whole equation, then immediately we can obtain (5.4b) And the reverse is also true.

Equation (5.4) corresponds to the integrability condition of the AKNS equation (i.e. the Dirac equation). The AKNS equation is linear equation

$$i\frac{\partial \phi}{\partial t}=\hat{N}\phi \quad \text{(time-evolution eq.)}$$

$$\hat{M}\phi=\zeta\gamma_4\phi \quad \text{(eigen-eg.)} \qquad (5.6)$$

where

$$\hat{M} = \begin{bmatrix} i\sigma_k\partial_k & -iu \\ -iv & i\sigma_k\partial_k \end{bmatrix} \quad (k=1, 2, 3) \left.\begin{array}{c}\\\\\\\\\\\end{array}\right\}$$

$$\hat{N} = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \qquad\qquad\qquad\qquad (5.7)$$

$\sigma_k (k=1, 2, 3)$ are the Pauli matrices, $\gamma_4=\sigma_3\otimes\sigma_4$ is the Dirac matrix, σ_4 is 2×2 unit matrix, ζ is eigenvalues and $\partial\zeta/\partial t=0$. In (5.7) and in the following, every quantity directly involves 2×2 unit matrix on convention, and in these quantities there doesn't exist the Pauli matrices σ_k. In addition, $\otimes$ is the symbol for direct product (or Krönecker product).

If A, B and C are

$$\begin{cases} A=2\zeta^2+vu \\ B=2iu\zeta-\sigma_k\partial_k u \qquad (k=1, 2, 3) \\ C=2iv\zeta+\sigma_k\partial_k v \end{cases} \qquad (5.8)$$

then the integrability condition of the AKNS equation (5.6) is

$$i\frac{\partial u}{\partial t} + \nabla^2 u - 2uvu = 0 \qquad (5.9a)$$

$$i\frac{\partial v}{\partial t} - \nabla^2 v + 2vuv = 0 \qquad (5.9b)$$

Under this condition, u and v are the real functions, we exchange u and v in equation (5.9a), and take the complex conjugate for the whole equation, then immediately we can obtain (5.9b). And the reverse is also true.

If we analytically continue the related functions in the AKNS equation (5.6) into the complex plane, in other words, if we turn the energy in equation (5.6) into a complex one, then (5.9) is the same as (5.4). Thus, the nonlinear Boltzmann differential-integral equation can finally be summed up as the solving of the linear Dirac differential-integral equation.

The meaning for complex energy as the above exposition, shows that the linear Dirac differential-integral equation is also irreversible. Look out, in this irreversible equation as equations (4.15) and (5.6), the dissipation coefficient[59] always equals one.

If the external field force exists in our problem, then the Virial for external field force can be summarized into equation (5.6) by the classical method.

Except for orthodox application in quantum electrodynamics, the Dirac equation unites[4] the KdV equation, the nonlinear Schrödinger equation, the sine-Gordon equation, the Bloch equation, the Lamb equation and the mutual action equation for three waves in the theory of nonlinear waves. In addition, the Dirac equation is applied to solve the Navier-Lamé equation[60-61] in elastic mechanics, the Love-Kirchhoff equation[62, 46] the von Kármán equation[11, 45, 47] and the von Kármán-Vlasov equation[46] in the theory of elastic plates and shells, and the dispersion problems and so on. Now, we introduce the complex energy, and extend the applicable range of the Dirac equation into the irreversible problem of this kind of solving the Boltzmann equation in the gas-kinetic theory. The profound intension for the Dirac equation becomes acceptable gradually.

References

[1] Ablowitz, M.J., D.J. Kaup, A.C. Newell and H. Segur, Method for solving the sine-Gordon equation, *Phys. Rev. Letters.*, **30** (1973), 1262-1246.

[2] Ablowitz, M.J., D.J. Kaup, A.C. Newell and H. Segur, Nonlinear evolution equations of physical significance, *Phys. Rev. Letters.*, **31** (1973), 125-127.

[3] Eckhans, W. and A. van Harten, *The Inverse Scattering Transformation and the Theory of Solitions*, North-Holland, Amsterdam (1981).

[4] Taniuti, T. and K. Nishihara, *Nonlinear Waves*, Iwanami, (1977); Pitman (1983).

[5] Zakharov, V.E., S.V. Manakov, S.P. Novikov and L.P. Pitaevskill, *Theory of Solition (Method for Inverse Problems)*, Physico-Math. (1980). (in Russian)

[6] Dirac, P.A.M., *The Principles of Quantum Mechanics*, Oxford (1958).

[7] Dirac, P.A.M., *Directions in Physics*, John Wiley (1978).

[8] Landau, L.D. and E.M. Lifshitz, *Course of Theoretical Physics*, vol. 4, (2nd ed.) Quantum Electrodynamics, (ed. by V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii) Trans. from Russian by J.B. Sykes, and J.S. Bell, Pergamon (1982).

[9] Feynman, R.P., Quamtum electrodynamics, *A Lecture Note and Peprint Volume*, W.A. Benjamin Inc. (1962).

[10] Bjorken, J.D. and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill (1964).

[11] Shen Hui-chuan, Again on the relation of von Kármán equation for elastic large deflection problem and Schrödinger equation for quantum eigenvalues problem, *Appl. Math. and Mech.*

[12] Haken,H., Synergetics-the self-tissue process in physics, chemology and biology in Chinese), Trans. from German by Zhu Z.H., *Physics*, **14**, 10 (1985), 604-613.

[13] Boltzmann, L., Weitere studien uber das wärmegleichgewicht unter gasmolekülen, wien. Ber., **66** (1972); **81** (1880); **84** (1881).

[14] Tsien Hsue-shen, *Equations of Gas Dynamics, Fundamentals of gas dynamics* (ed. by Emmons. H.W.) section A, Oxford University Press (1958).

[15] Ladouzhenskaia, O.A., *Mathematical Problems of Dynamics for Viscous Incompressible Fluid*, Science, Moscow (1970). (in Russian)

[16] Chapman, S. and T.G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (3rd ed.), Cambridge Univ. Press (1970).

[17] Landu, L.D., and E.M. Lifshitz, *Course of Theoretical Physics*, vol. **5**, Statistical Physics, pergamon. Oxford (1969).

[18] Landau, L.D. and E.M. Lifshitz, *Course of Theoretical Physics*, vol. **10**, Physical Kinematics, (ed. by E.M. Lifshitz, and L.P. Pitaevskii), Pergamon (1982); The Japanese ed., trans. from Russian by Inoue. K., Inshihashi. Z., and S. Yanagishita, Tokyo (1982).

[19] Yukawa, Hideki, *Basic of Theoretical Physics*, (2nd ed.), vol. **5**, Statistical Physics (ed. by Toda. S., and R. Hisayasu) Iwanami (1978). (in Japanese)

[20] Wu Da-you, *Theoretical Physics*, vol. **5**, Thermodynamics, Gas − kinetic Theory and Statistical Mechanics, Science (1983). (in Chinese)

[21] ter Haar, D., *Elements of Statistical Mechanics*, Rinehart, New York (1954).

[22] Wang Zhu-xi, *Introduction to Statistical Physics*, People-Education (1956). (in Chinese)

[23] Prigogine, I., *Non-Equilibrium Statistical Mechanics*, Interscience Publishers (1962).

[24] de Groot, S.R. and P. Mazur, *Non-Equilibrium Thermodynamics*, North-Holland, Company-Amsterdam (1962).

[25] Cheng Kai-jia, *Solid-State Physics*, People-Education (1959). (in Chinese)

[26] Boltzmann, L., *Vorlesungen über gastheorie*, Leipzig (1896–1898).

[27] Boltzmann, L., *Lectures on Gas Theory*, University of California Press, Berkeley (1964).

[28] Hilbert, D., *Grundzuge einer allgemeinen theorie der Linearen integralgleichungen*, Teubner (1912), 269.

[29] Zubarev, D.N., *Non-Equilibrium Statistical Thermodynamics*, Physco-Math., Moscow (1971). (in Russian)

[30] Grad, H., On the kinetic theory of rarefied gases, *Commum. on Pure and Appl. Math.*, **2** (1949), 331–407.

[31] Grad, H., Solution of the Boltzmann equation in an unbounded domain, *Commum. on Pure and Appl. Math.*, **18** (1965), 345–354.

[32] Grad, H., Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equation, *proc. Amer, Math. soc. Symp. on Appl. Math.*, **17**, AMS. Providence. RI., (1965), 154–183.

[33] Kihara, T., *Imperfect Gases*, Asakura Bookstore, Tokyo, (1949). English translation by U.S. Office of Air Research, Wright-Patterson Air Base.

[34] Waldmann, L., *Proc. Int. Seminar on Transport Properties of Gases*, Brown Univ. Providence. RI. (1964).

[35] Balescu, R., *Statistical Mechanics of Charged particles*, Interscience, New York, to appear (1963).

[36] Hirschfelder, J.O., C.F. Curtiss and R.B. Bird, *The Molecular Theory of Gases and Liquids*, with the assistance of the staff of the univ. of Wisconsin Naval Research Laboratory, Wiley, New York (1954).

[37] Cercignani, C., (ed.) *Kinetic Theories and the Boltzmann Equation*, Scottish Acad. Press, Edinburgh (1975).

[38] Cercignani, C., *Mathematieal Methods in Kinetic Theory*, Plenum (1969).

[40] Cercignani, C., Solution of the Boltzmann equation, *The Boltzmann Equation*, Ch.4, Studies in Statistical Mechanics, Vol. **10**, (ed. by E.W. Montroll, and J.L.Lebowitz), North-Holland (1983).

[41] Ernst, M.H., Exact solutions of the nonlinear Boltzmann equation and related kinetic equations, *The Boltzmann Equation*, Ch.3, Studies in Statistical Mechanics, vol. **10** (ed. by E.W.Montroll, and J.L. Lebowitz), North-Holland (1983).

[42] Goodman, F.D. and H.Y. Wachman, *Dynamics of Gas-Surface Scattering*, Acad. Press (1976).

[43] Lebowitz, J.L. and E.W. Montroll, Nonequilibrium phenomena I. *The Boltzmann Equation, Studies in Statistical Mechanics*, vol. **10**, (ed. by E.W. Montroll, and J.L. Lebowitz.), North-Holland (1983).

[44] haken, H., *Synergetics*, (2nd ed.), Springer (1979).

[45] Shen Hui-chuan, The relation of von Kármán equation for elastic large deflection problem and Schrödinger equation for quantum eigenvalues problem, *Appl. Math. and Mech.*, **6**, 8 (1985), 761–775.

[46] Shen Hui-chuan, The Schrödinger equation of thin shell theories, *Appl. Math. and Mech.*, **6**, 10 (1985), 957–973.

[47] Shen Hui-chuan, The Schrödinger Equation in theory of plates and shells with orthorhombic anisotropy, *Appl. Math. and Mech.*

[48] Ehrenfest, P. and T., *Enzyklopadie der mathematischen Wissenschaften*, vol. IV, pt. 32,

Leipzig-Berlin (1911), 36.

[49] Jeans, J.H., *Dynamical Theory of Gases*, Cambridge Univ press, Cambridge (1921).

[50] Born, M. and H.S. Green, *A General Kinetic Theory of Liquids*, Cambridge Univ. Press, Cambridge (1949).

[51] Bogoliubov, N.N., *problems of a Dynamical Theory in Statistical Physics*, state Technical press, Moscow, (1946), English translation by E.K. Gora, in Studies in Statistical Mechanics, vol. 1, (ed. by de Boer. J., and G.E. Uhlenbeck) (1962).

[52] Green, H.S., *The Molecular Theory of Fluids*, North-Holland, Amsterdam (1952).

[53] Kirkwood, J.G. and J. Ross, *proceedings of the International Symposium on Transport Processes in Statistical Mechanics*, Brussels (1965); Inter. Science, New York (1958).

[54] Yvon, J., *La Theorie des Fluids et de l'Equation d'etat*, Hermann et cie, paris (1935).

[55] Courant, R. and D.Hilbert, *Methoden der Math. Phys.*, Julius. Springer (1931 – 1937).

[56] Landau, L.D. and E.M. Lifshitz, *Course of Theoretical Physics, vol.* 3, Quantum Mechanics (non-relativistic theory), 3rd ed., Translated from Russian by Sykes. J.B., and J.S. Bell, Pergamon (1977).

[57] Goldstein, H., *Classical Mechanics*, Addison-Wesley (1953).

[58] Greenwood, D.T., *Classical Dynamics*, Prentice-Hall (1977).

[59] Shen Hui-chuan, Dissipation mechanics and exact solutions for nonlinear equations of dissipative type-Principle and application of dissipation mechanics (I), *Appl. Math. and Mech.* 7, 12 (1986).

[60] Shen Hui-chuan, General solution of elastodynamics, *Appl. Math. and Mech.*, 6, 9 (1985), 853 – 858.

[61] Shen Hui-chuan, The fission of spectrum line of monochoromatic elastic wave, *Appl. Math. and Mech.*, 5, 4 (1984), 1509 – 1619.

[62] Shen Hui-chuan, The solution of deflection of elastic thin plate by the joint action of dynamical lateral pressure, force in central surface and external field on the elastic base, *Appl. Math. and Mech.*, 5, 6 (1984), 1791 – 1801.