

ON THE JUMPING PROBLEMS OF A CIRCULAR THIN PLATE WITH INITIAL DEFLECTION

Qin Sheng-li (秦圣立) and Zhang Ai-shu (张爱淑)

(Department of Physics, The Teachers' College of Qufu, Qufu, Shandong)

(Received June 2, 1986 Communicated by Jiang Fu-ru)

Abstract

In this paper, the jumping problems of a circular thin plate with initial deflection are studied by using the method of two variables^{[3],[4]} proposed by Jiang Fu-ru and the method of the normal perturbation (in this paper (1.1), (1.2)). We obtain Nth-order uniformly valid asymptotic expansion of the solution of this problem ((1.66), (1.67)). When the initial deflection vanishes the solution of a circular thinplate with initial deflection is reduced to the solution of the problems of the nonlinear bending of a circular thin plate^[6]. If the initial deflection is largish and the signs of the initial deflection with the intensity of the transverse load are opposite, when the intensity of the transverse load reaches a certain value, the circular thin plate with initial deflection should produce the jumping phenomenon^[8].

As early as 1948, Chien Wei-zang^[1] studied the nonlinear bending problem for a clamped circular plate under uniform normal pressure, and derived the asymptotic solution of the bending plate, which is in agreement with experiment and laid a favorable foundation in the study of large deflection of thin plates by using membrane solution.

In 1982, Jiang Fu-ru^[2] studied the nonlinear and unsymmetrical bending problems of annular and circular thin plates under various supports by means of the method presented in [3] and [4]. Making use of the membrane solution, he constructed the formal asymptotic solution of these bending problems up to Nth-order.

In this paper, we study the jumping problems of a circular thin plate with initial deflection by using the method presented in [3] and [4] and the method of the normal perturbation^[7].

I. The Circular Thin Plate with Initial Deflection.

For the circular thin plate with initial deflection under uniform normal pressure if its edge is clamped, we know the deflection function W and stress function Φ of the bending plate are satisfied by the following equations and the boundary conditions^[8]:

$$\left. \begin{aligned} D \frac{d}{dr} [\nabla^2 W] &= \psi + \frac{h}{r} \frac{d\Phi}{dr} \left(\frac{dW}{dr} + \frac{dW_{IN}}{dr} \right) \\ \frac{d}{dr} [\nabla^2 \Phi] &= - \frac{E}{r} \left[\frac{1}{2} \left(\frac{dW}{dr} \right)^2 \frac{dW_{IN}}{dr} \frac{dW}{dr} \right] \end{aligned} \right\} \quad (1.1)$$

$$\left. \begin{aligned} W|_{r=0} &= 0, & W_{,r}|_{r=c} &= 0, & W_{,r}|_{r=0} &= 0 \\ \frac{d\Phi}{dr}\Big|_{r=0} &= 0, & \left(\frac{d^2\Phi}{dr^2} - \frac{\mu}{r} \frac{d\Phi}{dr} \right)\Big|_{r=0} &= 0 \end{aligned} \right\} \quad (1.2)$$

Where, E is modulus of elasticity, h is the thickness of plate, $D = Eh^3/12(1-\mu^2)$ is the flexural rigidity, μ is Poissons' ratio, q is the intensity of the normal pressure, ∇^2 is the two-dimensional Laplace operator in polar coordinates (r, θ) , and W_{IN} is initial deflection function, we suppose that the initial deflection function is:

$$W_{IN} = \bar{W}_0 \left(1 + \cos \frac{\pi}{c} r \right) \quad (1.3)$$

Where \bar{W}_0 is one-half of the maximum value of the initial deflection, c is radius of the circular thin plate; and Ψ is the loading function:

$$\begin{aligned} \Psi &= \frac{1}{2\pi r} \int_{\mathcal{G}} q \, ds \\ &= \frac{q}{\pi} \int_0^r r \left[1 + \frac{\pi^2 \bar{W}_0^2}{c^2} \sin^2 \left(\frac{\pi}{c} r \right) \right]^{1/2} dr \end{aligned} \quad (1.4)$$

Let $\bar{W}_0/c = \delta \ll 1$, and we shall expand $[1/\pi^2 + \delta^2 \sin^2(\pi r/c)]^{1/2}$ into the power series of the $\delta^2 \sin^2(\pi r/c)$, and integral (1.4), we obtain:

$$\begin{aligned} \Psi &= \frac{qr}{2} + \frac{q}{8r} \delta^2 \left[\pi^2 r^2 - c\pi r \sin \left(\frac{2\pi}{c} r \right) \right] \\ &\quad + \frac{c^2}{2} \left(1 - \cos \left(\frac{2\pi}{c} r \right) \right) + O(\delta^4) \end{aligned} \quad (1.5)$$

We substitute (1.3) and (1.5) into equation (1.1), and can write them in dimensionless form as:

$$\left. \begin{aligned} \varepsilon^2 \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) \right] &= \frac{qr}{2} + \frac{q}{8r} \delta^2 \left[\pi^2 cr^2 - \pi cr \sin 2\pi r + \frac{c}{2} (1 - \cos 2\pi r) \right] \\ &\quad + \frac{1}{r} \frac{d\Phi}{dr} \left(\frac{dW}{dr} - \pi \delta \sin \pi r \right) + O(\delta^4) \\ \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\Phi}{dr} \right) \right] &= -\frac{1}{r} \left[\frac{1}{2} \left(\frac{dW}{dr} \right)^2 - \pi \delta \sin \pi r \right] \end{aligned} \right\} \quad (1.6)$$

Where $\varepsilon^2 = h^2/12(1-\mu^2)c^2 \ll 1$

The boundary conditions (1.2) are reduced to the following form:

$$\left. \begin{aligned} W|_{r=1} &= 0, & W_{,r}|_{r=1} &= 0, & W_{,r}|_{r=0} &= 0 \\ \frac{d\Phi}{dr}\Big|_{r=0} &= 0, & \left(\frac{d^2\Phi}{dr^2} - \frac{\mu}{r} \frac{d\Phi}{dr} \right)\Big|_{r=1} &= 0 \end{aligned} \right\} \quad (1.7)$$

In order to correct boundary conditions, we first expand the differential operators^{[3][4]}. In the neighborhood of edge $r=1$, introduce two variables ξ and η :

$$\xi = \frac{u(r)}{e}, \quad \eta = r$$

By transforming the partial derivatives with respect to r into partial derivatives with respect to ξ and η , and regard ξ , η as independent variables^[5].

$$\left. \begin{aligned} \frac{d}{dr} &= \varepsilon^{-1} \left(u_r \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \eta} \right) \\ &\dots \dots \dots \end{aligned} \right\} \quad (1.8)$$

In order to obtain the recursive equations and boundary conditions, we suppose that the N th-order asymptotic approximations of deflection and stress function are

$$\left. \begin{aligned} W(r, \varepsilon) &= \sum_{n=0}^N \varepsilon^n W_n(r) + \sum_{n=0}^N \varepsilon^{n+\alpha} v_n(\xi, \eta) \\ \Phi(r, \varepsilon) &= \sum_{n=0}^N \varepsilon^n \varphi_n(r) + \sum_{n=0}^N \varepsilon^{n+\beta} \psi_n(\xi, \eta) \end{aligned} \right\} \quad (1.9)$$

Upon substituting (1.8) and (1.9) into equations (1.6) and boundary conditions (1.7), and considering the properties of the boundary layer functions, and we note that φ_n , w_n ($n=0, 1, 2, \dots, N$) are functions of r only. In order to obtain recursive equations and boundary conditions, comparing the coefficient of the lowest powers of ε^n , we see that we should take $\alpha=1$, $\beta=3$ ^[3]. then comparing the coefficients of ε^n , we obtain the recursive equations and boundary conditions of w_n , v_n , φ_n , ψ_n ($n=0, 1, \dots, N$):

$$\begin{aligned} \frac{1}{r} \frac{d\varphi_0}{dr} \left[\frac{dw_0}{dr} - \pi \delta \sin \pi r \right] &= - \left[\frac{qr}{2} + \frac{q}{8r} \delta^2 (\pi^2 cr^2 \right. \\ &\quad \left. - \pi cr \sin 2\pi r + \frac{c}{2} - \frac{c}{2} \cos 2\pi r) \right] \end{aligned} \quad (1.10)$$

$$\frac{d}{dr} \left[\frac{1}{r} \left(r \frac{d\varphi_0}{dr} \right) \right] + \frac{1}{r} \left[\frac{1}{2} \left(\frac{dw_0}{dr} \right)^2 - \pi \delta \frac{dw_0}{dr} \sin \pi r \right] = 0 \quad (1.11)$$

$$\begin{aligned} \frac{1}{r} \frac{d\varphi_n}{dr} \frac{dw_0}{dr} - \frac{1}{r} \pi \delta \frac{d\varphi_n}{dr} \sin \pi r + \frac{1}{r} \frac{d\varphi_0}{dr} \frac{dw_n}{dr} \\ = \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw_{n-2}}{dr} \right) \right] - \frac{1}{r} \sum_{i=1}^{n-1} \frac{d\varphi_i}{dr} \frac{dw_{n-i}}{dr} \end{aligned} \quad (1.12)$$

$$\begin{aligned} \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_n}{dr} \right) \right] - \frac{1}{r} \pi \delta \frac{dw_n}{dr} \sin \pi r \\ + \frac{1}{r} \frac{dw_0}{dr} \frac{dw_n}{dr} = - \frac{1}{2r} \sum_{i=1}^{n-1} \left(\frac{dw_i}{dr} \frac{dw_{n-i}}{dr} \right) \end{aligned} \quad (1.13)$$

$$D_0 v_0 - \frac{u_r}{\eta} \frac{\partial \varphi_0}{\partial \eta} \frac{\partial v_0}{\partial \xi} = 0 \quad (1.14)$$

$$\begin{aligned} D_0 \psi_0 + \frac{u_r^2}{2\eta} \left(\frac{\partial v_0}{\partial \xi} \right)^2 + \frac{u_r}{\eta} \frac{\partial w_0}{\partial \eta} \frac{\partial v_0}{\partial \xi} \\ - \pi \delta \frac{u_r}{\eta} \frac{\partial v_0}{\partial \xi} \sin \pi \eta = 0 \end{aligned} \quad (1.15)$$

$$\begin{aligned}
D_0 v_n - \frac{u_r}{\eta} \frac{\partial \varphi_0}{\partial \eta} \frac{\partial v_n}{\partial \xi} = & - \sum_{i=1}^3 D_i v_{n-i} \\
& + \frac{v_r}{\eta} \sum_{i=1}^n \frac{\partial \varphi_i}{\partial \eta} \frac{\partial v_{n-i}}{\partial \xi} + \frac{1}{\eta} \sum_{i=0}^n \frac{\partial \varphi_i}{\partial \eta} \frac{\partial v_{n-1-i}}{\partial \eta} \\
& + \frac{u_r^2}{\eta} \sum_{i=0}^n \frac{\partial \psi_i}{\partial \xi} \frac{\partial v_{n-2-i}}{\partial \xi} + \frac{u_r}{\eta} \sum_{i=0}^n \frac{\partial \psi_i}{\partial \xi} \frac{\partial v_{n-3-i}}{\partial \eta} \\
& + \frac{u_r}{\eta} \sum_{i=0}^n \frac{\partial \psi_i}{\partial \eta} \frac{\partial v_{n-3-i}}{\partial \xi} + \frac{1}{\eta} \sum_{i=0}^n \frac{\partial \psi_i}{\partial \eta} \frac{\partial v_{n-4-i}}{\partial \eta} \\
& + \frac{u_r}{\eta} \sum_{i=0}^n \frac{\partial \psi_{n-2-i}}{\partial \xi} \frac{\partial w_i}{\partial \eta} + \frac{1}{\eta} \sum_{i=0}^n \frac{\partial \psi_{n-3-i}}{\partial \eta} \frac{\partial w_i}{\partial \eta} \\
& - \frac{1}{\eta} \pi \delta \frac{\partial \psi_{n-2}}{\partial \xi} - \frac{1}{\eta} \pi \delta \frac{\partial \psi_{n-3}}{\partial \eta}
\end{aligned} \quad (1.16)$$

$$\begin{aligned}
D_0 \psi_n = & - \sum_{i=1}^3 D_i \psi_{n-i} - \frac{u_r^2}{2\eta} \sum_{i=0}^n \frac{\partial v_i}{\partial \xi} \frac{\partial v_{n-i}}{\partial \xi} \\
& - \frac{1}{2\eta} \sum_{i=0}^n \frac{\partial v_i}{\partial \eta} \frac{\partial v_{n-2-i}}{\partial \eta} - \frac{u_r}{\eta} \sum_{i=0}^n \frac{\partial v_i}{\partial \eta} \frac{\partial v_{n-1-i}}{\partial \eta} \\
& - \frac{u_r}{\eta} \sum_{i=0}^n \frac{\partial w_i}{\partial \eta} \frac{\partial v_{n-i}}{\partial \xi} - \frac{1}{\eta} \sum_{i=0}^n \frac{\partial w_i}{\partial \eta} \frac{\partial v_{n-1-i}}{\partial \eta} \\
& + \frac{u_r}{\eta} \pi \delta \frac{\partial v_n}{\partial \xi} \sin \pi \eta + \frac{1}{\eta} \pi \delta \frac{\partial v_{n-1}}{\partial \eta} \sin \pi \eta
\end{aligned} \quad (1.17)$$

and the boundary conditions for w , v , φ , ψ ($n=0,1,2,\dots, N$) are:

$$w_0|_{r=1}=0 \quad (1.18)$$

$$w_{0,r} \Big|_{r=1} + u_r \frac{\partial v_0}{\partial \xi} \Big|_{\eta=1} = 0 \quad (1.19)$$

$$w_{0,r} \Big|_{r=0} + u_r \frac{\partial v_0}{\partial \xi} \Big|_{\eta=0} = 0 \quad (1.20)$$

$$\varphi_{0,r}|_{r=0}=0 \quad (1.21)$$

$$\varphi_{0,rr} \Big|_{r=1} - \frac{\mu}{r} \varphi_{0,r} \Big|_{r=1} = 0 \quad (1.22)$$

$$w_n|_{r=1} + v_{n-1}|_{\eta=1}=0 \quad (1.23)$$

$$w_{n,r} \Big|_{r=1} + u_r \frac{\partial v_n}{\partial \xi} \Big|_{\eta=1} + \frac{\partial v_{n-1}}{\partial \eta} \Big|_{\eta=1} = 0 \quad (1.24)$$

$$w_{n,r} \Big|_{r=0} + u_r \frac{\partial v_n}{\partial \xi} \Big|_{\eta=0} + \frac{\partial v_{n-1}}{\partial \eta} \Big|_{\eta=0} = 0 \quad (1.25)$$

$$\varphi_{n,r} \Big|_{r=0} + u_r \frac{\partial \psi_{n-2}}{\partial \xi} \Big|_{\eta=0} + \frac{\partial \psi_{n-3}}{\partial \eta} \Big|_{\eta=0} = 0 \quad (1.26)$$

$$\begin{aligned} \varphi_{n,rr} \Big|_{r=1} - \frac{\mu}{r} \varphi_{n,r} \Big|_{r=1} + u_r^2 \frac{\partial^2 \psi_{n-1}}{\partial \xi^2} \Big|_{\eta=1} \\ + \left(2u_r \frac{\partial^2}{\partial \xi \partial \eta} + u_{rr} \frac{\partial}{\partial \xi} \right) \psi_{n-2} \Big|_{\eta=1} + \frac{\partial^2 \psi_{n-3}}{\partial \eta^2} \Big|_{\eta=1} \\ - \frac{\mu u_r}{\eta} \frac{\partial \psi_{n-2}}{\partial \xi} \Big|_{\eta=1} - \frac{\mu}{\eta} \frac{\partial \psi_{n-3}}{\partial \eta} \Big|_{\eta=1} = 0 \end{aligned} \quad (1.27)$$

where

$$\begin{aligned} D_0 &= u_r^3 \frac{\partial^3}{\partial \xi^3}, \\ D_1 &= 3u_r^2 \frac{\partial^3}{\partial \xi^2 \partial \eta} + \left(\frac{u_r^2}{\eta} + 3u_r u_{rr} \right) \frac{\partial^2}{\partial \xi^2}, \\ D_2 &= \left(\frac{2u_r}{\eta} + 3u_{rr} \right) \frac{\partial^2}{\partial \xi \partial \eta} + 3u_r \frac{\partial^3}{\partial \xi \partial \eta^2} + \left(\frac{u_{rr}}{\eta} + u_{rrr} - \frac{u_r}{\eta^2} \right) \frac{\partial}{\partial \xi}, \\ D_3 &= \frac{\partial^3}{\partial \eta^3} + \frac{1}{\eta} \frac{\partial^2}{\partial \eta^2} - \frac{1}{\eta^2} \frac{\partial}{\partial \eta}. \end{aligned}$$

From now on, the letters with negative subscripts are taken as zero.

In the following, we shall find for the formal asymptotic solutions of the deflection and stress function W , Φ

From equations (1.0) and (1.11), and boundary conditions (1.18), (1.21) and (1.22), and by using the method of the normal perturbation, we can determine w_0 and $d\varphi_0/dr$

We suppose that the M th-order asymptotic expansions of w_0 and $d\varphi_0/dr$ are

$$\begin{cases} w_0(r, \delta) = \sum_{n=0}^M \delta^n w_{0n} \end{cases} \quad (1.28a)$$

$$\begin{cases} \frac{d\varphi_0(r, \delta)}{dr} = \sum_{n=0}^M \delta^n \frac{d\varphi_{0n}}{dr} \end{cases} \quad (1.28b)$$

Substituting (1.28) into equations (1.10) and (1.11), and comparing the coefficients of δ^n , equating the coefficients of δ^0 to $-qr/2$, and equating the coefficients of δ to each other, respectively, we obtain the recursive equations for w_{0n} , φ_{0n} , ($n=0, 1, \dots, M$):

$$\frac{1}{r} \frac{d\varphi_{00}}{dr} \frac{dw_{00}}{dr} = -\frac{qr}{2} \quad (1.29)$$

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_{00}}{dr} \right) \right] = -\frac{1}{2r} \left(\frac{dw_{00}}{dr} \right)^2 \quad (1.30)$$

$$\frac{1}{r} \frac{d\varphi_{00}}{dr} \left(\frac{dw_{01}}{dr} - \pi \sin \pi r \right) + \frac{1}{r} \frac{d\varphi_{01}}{dr} \frac{dw_{00}}{dr} = 0 \quad (1.31)$$

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_{01}}{dr} \right) \right] + \frac{1}{r} \left[\frac{dw_{00}}{dr} \frac{dw_{01}}{dr} - \pi \frac{dw_{00}}{dr} \sin \pi r \right] = 0 \quad (1.32)$$

$$\frac{1}{r} \frac{d\varphi_{02}}{dr} \frac{dw_{00}}{dr} + \frac{1}{r} \frac{d\varphi_{01}}{dr} \left(\frac{dw_{01}}{dr} - \pi \sin \pi r \right) + \frac{1}{r} \frac{d\varphi_{00}}{dr} \frac{dw_{02}}{dr} = -\frac{q}{8r} \left[\pi^2 cr^2 - \pi cr \sin 2\pi r + \frac{c}{2} (1 - \cos 2\pi r) \right] \quad (1.33)$$

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_{02}}{dr} \right) \right] + \frac{1}{r} \frac{dw_{02}}{dr} \frac{dw_{00}}{dr} + \frac{1}{2r} \left(\frac{dw_{01}}{dr} \right)^2 - \frac{\pi}{r} \frac{dw_{01}}{dr} \sin \pi r = 0 \quad (1.34)$$

.

Upon substituting (1.28) into boundary conditions (1.18), (1.21) and (1.22) and comparing the coefficients of the δ^n ($n=0, 1, 2, \dots, M$), equating the coefficients of the same powers of δ^n to zero, we then obtain the boundary conditions for w_{0n} , φ_{0n} ($n=0, 1, 2, \dots, M$):

$$w_{00}|_{r=1} = 0 \quad (1.35)$$

$$\left. \frac{d\varphi_{00}}{dr} \right|_{r=0} = 0 \quad (1.36)$$

$$\varphi_{00,rr} \Big|_{r=1} - \frac{\mu}{r} \varphi_{00,r} \Big|_{r=1} = 0 \quad (1.37)$$

$$w_{0n}|_{r=1} = 0 \quad (1.38)$$

$$\varphi_{0n,r} \Big|_{r=0} = 0 \quad (1.39)$$

$$\varphi_{0n,rr} \Big|_{r=1} - \frac{\mu}{r} \varphi_{0n,r} \Big|_{r=1} = 0 \quad (1.40)$$

From equations (1.29), (1.30) and boundary conditions (1.35)–(1.37), we can determine w_{00} and $\frac{d\varphi_{00}}{dr}$ by means of the solving method of power series, we obtain $\frac{d\varphi_{00}}{dr}$:

$$\frac{d\varphi_{00}}{dr} = a_1 r + a_3 r^3 + \dots \quad (1.41)$$

where

$$a_1 = \frac{1}{4} \left(\frac{3-\mu}{1-\mu} |q^2| \right)^{1/3}, \quad a_3 = -\frac{1}{4} \left(\frac{1-\mu}{3-\mu} q \right)^{2/3}$$

Substituting (1.41) into equation (1.29) and integrate this equation, and from boundary condition (1.35), we get:

$$w_{00} = -\frac{q}{4a_3} \ln \left(\frac{1 - (1-\mu)r^2/(3-\mu)}{1 - (1-\mu)/(3-\mu)} \right) \quad (1.42)$$

Substituting (1.41) and (1.42) into equations (1.31) and (1.32), and from boundary conditions (1.38), (1.39) and (1.40) (respectively take $n=1$), we obtain:

$$\frac{d\varphi_{01}}{dr} = 0 \quad (1.43)$$

$$w_{01} = -(1 + \cos \pi r) \quad (1.44)$$

We substitute (1.43) and (1.44) into equations (1.33) and (1.34), and from boundary conditions (1.38)–(1.40) (respectively take $n=2$), hence we can determine w_{02} and $d\varphi_{02}/dr$:

$$\frac{d\varphi_{02}}{dr} = e_1 r + e_3 r^3 + e_5 r^5 + \dots \quad (1.45)$$

$$\begin{aligned} w_{02} = & -\frac{q\pi^2 c}{8} \left[\frac{1}{2a_3} \ln \left(\frac{a_1 + a_3 r^3}{a_1 + a_3} \right) \right] + \frac{q\pi c}{8} \int_1^r \frac{\sin 2\pi r}{a_1 + a_3 r^2} dr \\ & + \frac{qc}{16} \int_1^r \frac{(\cos 2\pi r - 1)}{r(a_1 + a_3 r^2)} dr + \frac{q}{2} \int_1^r \frac{r(e_1 + e_3 r^2 + e_5 r^4)}{(a_1 + a_3 r^2)^2} dr \end{aligned} \quad (1.46)$$

where

$$\begin{aligned} e_1 = & \frac{-\pi^4/16}{(1-\mu) + (3-\mu)\frac{q^3}{32a_1^3} + \frac{(5-\mu)q^2(q^2-96a_1^2a_3)}{3072a_1^6}} \left\{ (3-\mu) \right. \\ & \left. + \frac{(5-\mu)(q^2-96a_1^2a_3)}{96a_1^3} + \frac{(5-\mu)}{72} \left(72\left(\frac{a_3}{a_1}\right) - \frac{3q_c^2}{a_1^2} - 8\pi^2 \right) \right\} \\ e_3 = & \frac{q^2}{32a_1^3} e_1 + \frac{\pi^4}{16} \\ e_5 = & \frac{1}{96a_1^3} (q^2 - 96a_1^2a_3) e_3 + \frac{\pi^4}{1152} \left(72\frac{a_3}{a_1} - \frac{3q_c^2}{a_1^2} - 8\pi^2 \right) \end{aligned}$$

According to the above steps, we can successively determine w_{0n} , $d\varphi_{0n}/dr$ ($n=0, 1, 2, \dots, M$), we confine ourselves to the approximate solutions of two-order for w_{0n} and $d\varphi_{0n}/dr$ ($n=0, 1, 2, \dots, M$), and substituting (1.41), (1.42), (1.43), (1.44), (1.45) and (1.46) into (1.28a) and (1.28b), then we obtain:

$$\begin{aligned} w_0(r, \delta) = & -\frac{q}{4a_3} \ln \left(\frac{1 - (1-\mu)r^2/(3-\mu)}{1 - (1-\mu)/(3-\mu)} \right) - \delta(1 + \cos \pi r) \\ & - \delta^2 \left[-\frac{q\pi^2}{16a_3} \ln \left(\frac{a_1 + a_3 r^3}{a_1 + a_3} \right) - \frac{qc}{16} \int_1^r \frac{(\cos 2\pi r - 1)}{r(a_1 + a_3 r^2)} dr \right. \\ & \left. - \frac{q\pi c}{8} \int_1^r \frac{\sin 2\pi r}{(a_1 + a_3 r^2)} dr - \frac{q}{2} \int_1^r \frac{r(e_1 + e_3 r^2 + e_5 r^4)}{(a_1 + a_3 r^2)^2} dr \right] + O(\delta^3) \end{aligned} \quad (1.47)$$

$$\frac{d\varphi_0(r, \delta)}{dr} = (a_1 r + a_3 r^3) + \delta^2(e_1 r + e_3 r^3 + e_5 r^5) + O(\delta^3) \quad (1.48)$$

After determining $d\varphi_0/dr$ and w_0 , and substituting (1.47) and (1.48) into (1.41), we obtain the differential equation for v_0 :

$$u_r^3 \frac{\partial^2 v_0}{\partial \xi^2} - \frac{u_r}{\eta} [(a_1 r + a_3 r^3) + \delta^2(e_1 r + e_3 r^3 + e_5 r^5)] \frac{\partial v_0}{\partial \xi} = 0 \quad (1.49)$$

If we choose

$$u_r^2 = [(a_1 r + a_3 r^3) + \delta^2(e_1 r + e_3 r^3 + e_5 r^5)]/\eta$$

then

$$u_r(r) = \int_r^1 [(a_1 + a_3 \eta^2) + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} d\eta \quad (1.50)$$

At this juncture, equation (1.49) is reduced to the homogeneous equation:

$$\frac{\partial^3 v_0}{\partial \xi^3} - \frac{\partial v_0}{\partial \xi} = 0 \quad (1.51)$$

From equation (1.51), we obtain the solution of the boundary layer type for v :

$$\begin{aligned} v_0 &= c_0(\eta) \exp[-\xi] \\ &= c_0(r) \exp \left[-\varepsilon^{-1} \int_r^1 [(a_1 + a_3 \eta^2) + \delta^2 (e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} d\eta \right] \end{aligned} \quad (1.52)$$

where $c_0(\eta)$ is arbitrary function to be determined later (See (1.60) and (1.61)). We substitute w_0 and v_0 into equation (1.15), then we obtain the differential equation for ψ_0 :

$$\begin{aligned} u_r^3 \frac{\partial^3 \psi_0}{\partial \xi^3} &= -\frac{u_r^2}{2\eta} c_0^2(\eta) \exp(-2\xi) - \left\{ \frac{u_r}{\eta} \left[\frac{q\eta}{2(a_1 + a_3 \eta^2)} \right. \right. \\ &\quad + \frac{q\delta^2}{8(a_1 \eta + a_3 \eta^3)} (\pi^2 c \eta^2 - \pi c \eta \sin 2\pi \eta + \frac{c}{2} (1 - \cos 2\pi \eta)) \\ &\quad \left. \left. - \frac{q\delta^2 (e_1 \eta + e_3 \eta^3 + e_5 \eta^5)}{2(a_1 + a_3 \eta^2)^2} \right] \right\} c_0(\eta) \exp(-\xi) \end{aligned} \quad (1.53)$$

From equation (1.53) we can obtain the solution of the boundary layer type for ψ_0 :

$$\begin{aligned} \psi_0 &= \frac{1}{16\eta u_r} c_0^2(\eta) \exp(-2\xi) \\ &\quad + \frac{1}{\eta u_r^2} \left\{ \frac{q\eta}{2(a_1 + a_3 \eta^2)} + \frac{q\delta^2}{8(a_1 \eta + a_3 \eta^3)} [\pi^2 c \eta^2 \right. \\ &\quad \left. - \pi c \eta \sin 2\pi \eta + \frac{c}{2} (1 - \cos 2\pi \eta) \right. \\ &\quad \left. - \frac{q\delta^2 (e_1 \eta + e_3 \eta^3 + e_5 \eta^5)}{2(a_1 + a_3 \eta^2)^2} \right\} c_0(\eta) \exp(-\xi) \end{aligned} \quad (1.54)$$

We substitute w_0 and $d\varphi_0/dr$ into equations (1.12) and (1.13) (respectively for $n=1$), we obtain the differential equations for w_1 and φ_1 , then first we eliminate dw_1/dr , and we ignore the terms of order δ^2 , we obtain the equation for φ_1 :

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_1}{dr} \right) \right] - \frac{q^2}{4(a_1 + a_3 r^2)^2} \frac{d\varphi_1}{dr} = 0 \quad (1.55)$$

From equation (1.55) and boundary conditions (1.26) and (1.27) (respectively for $n=1$). Similarly, by means of the solving method of power series, we can determine $d\varphi_1/dr$:

$$\frac{d\varphi_1}{dr} = b_1 r + b_3 r^3 + \dots \quad (1.56)$$

where

$$\begin{aligned} b_1 &= -\frac{1}{(1-\mu) + q^2(3-\mu)/32a_1^3} \left\{ \left[\frac{q}{2(a_1 + a_3)} + \frac{qc\delta^2(\pi^2 - 1)}{8(a_1 + a_3)} \right. \right. \\ &\quad \left. \left. - \frac{q\delta^2(e_1 + e_3 + e_5)}{2(a_1 + a_3)} \right] c_0(1) + \frac{u_r(1)c_0^2(1)}{4} \right\}, \\ b_3 &= \frac{q^2}{32a_1^3} b_1 \end{aligned}$$

Substituting (1.47), (1.48) and (1.56) into (1.12), and by means of boundary conditions (1.23) (for $n=1$), we get:

$$\begin{aligned} w_1 = & \frac{b_3 q}{4a_3^2} \ln\left(\frac{a_1 + a_3 r^2}{a_1 + a_3}\right) \\ & - \frac{q}{4a_3} \left(\frac{b_1 + b_3 r^2}{a_1 + a_3 r^2} - \frac{b_1 + b_3}{a_1 + a_3} \right) - c_0(1) \\ & + \frac{q\delta^2}{16} \int_1^r \frac{(b_1 + b_3 r^2) \left(\pi^2 c r^2 - \pi c r \sin 2\pi r + \frac{c}{2} (1 - \cos 2\pi r) \right)}{r(a_1 + a_3 r^2)^3} dr \\ & + \frac{q\delta^2}{2} \int_1^r \frac{e_1 r^2 + e_3 r^4 + e_5 r^6}{(a_1 + a_3 r^2)^2} dr \end{aligned} \quad (1.57)$$

For equation (1.16) (for $n=1$), we obtain the differential equation for v_1 :

$$D_\eta v_1 - \frac{u_r}{\eta} \frac{\partial \varphi_0}{\partial \eta} \frac{\partial v_1}{\partial \xi} = -D_1 v_0 + \frac{u_r}{\eta} \frac{\partial \varphi_1}{\partial \eta} \frac{\partial v_0}{\partial \xi} + \frac{1}{\eta} \frac{\partial \varphi_0}{\partial \eta} \frac{\partial v_0}{\partial \eta} \quad (1.58)$$

Expanding the right-hand side of equation (1.58), and equating the coefficient of $\exp(-\xi)$ to zero, then we obtain the differential equation for $c_0(\eta)$:

$$\begin{aligned} 2[(a_1 + a_3 \eta^2) + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)] & - \frac{\partial c_0(\eta)}{\partial \eta} + \frac{1}{\eta} \{ (a_1 + a_3 \eta^2) \\ & + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4) + 3[a_3 \eta^2 + \delta^2(e_3 \eta^2 + 2e_5 \eta^4)] \\ & + [(a_1 + a_3 \eta^2) + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} (b_1 \eta + b_3 \eta^3) \} c_0(\eta) = 0 \end{aligned} \quad (1.59)$$

From differential equation (1.59) and boundary condition (1.19), we can determine $c_0(\eta)$:

$$\begin{aligned} c_0(\eta) = c_0(1) \exp \left[\int_1^\eta A^{-1} \{ a_1 + 4a_3 \eta^2 + \delta^2(e_1 + 4e_3 \eta^2 + 7e_5 \eta^4) \right. \\ \left. + [(a_1 + a_3 \eta^2) + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} (b_1 \eta + b_3 \eta^3) \} d\eta \right] \end{aligned} \quad (1.60)$$

where

$$\begin{aligned} A = 2\eta[(a_1 + a_3 \eta^2) + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)] \\ c_0(1) = \frac{q \left[1 + \frac{\pi^2 c \delta^2}{4} - \frac{\delta^2(e_1 + e_3 + e_5)}{(a_1 + a_3)} \right]}{2(a_1 + a_3) [(a_1 + a_3) + \delta^2(e_1 + e_3 + e_5)]^{1/2}} \end{aligned} \quad (1.61)$$

We substitute (1.60) into (1.52), and introducing a differentiable cut-off function $\xi(r)$: $\xi(r)=1$ for $2/3 \leq r \leq 1$, $\xi(r)=0$ for $0 \leq r < 2/3$; and substituting (1.60) into (1.52), then we obtain v_0 :

$$\begin{aligned} v_0 = \xi(r) c_0(1) \exp \left[-\varepsilon^{-1} \int_r^1 [a_1 + a_3 \eta^2 + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} d\eta \right. \\ \left. + \int_r^1 A^{-1} \{ a_1 + 4a_3 \eta^2 + \delta^2(e_1 + 4e_3 \eta^2 + 7e_5 \eta^4) \right. \\ \left. + [a_1 + a_3 \eta^2 + \delta^2(e_1 + e_3 \eta^2 + e_5 \eta^4)]^{1/2} (b_1 \eta + b_3 \eta^3) \} d\eta \right] \end{aligned} \quad (1.62)$$

Where A is the same as that in equation (1.60).

At this juncture, differential equation (1.58) for v_1 is reduced to the following equation:

$$u_r^3 \frac{\partial^3 v_1}{\partial \xi^3} - \frac{u_r}{\eta} \frac{\partial \varphi_0}{\partial \eta} \frac{\partial v_1}{\partial \xi} = 0 \quad (1.63)$$

Similarly, we can find the solution of the boundary layer type for v_1 :

$$v_1 = c_1(\eta) \exp \left[-\varepsilon^{-1} \int_r^1 [a_1 + a_3 \eta^2 + \delta^2 (e_1 + e_3 \eta^2 + e_6 \eta^4)]^{\frac{1}{2}} d\eta \right] \quad (1.64)$$

where $c_1(\eta)$ is arbitrary function to be determined later.

From equation (1.17) (for $n=1$), then we obtain the differential equation for ψ_1 :

$$\begin{aligned} D_0 \psi_1 = & -D_1 \psi_0 - \frac{u_r^2}{\eta} \frac{\partial v_0}{\partial \xi} \frac{\partial v_1}{\partial \xi} - \frac{u_r}{\eta} \frac{\partial v_0}{\partial \xi} \frac{\partial v_0}{\partial \eta} \\ & - \frac{u_r}{\eta} \frac{\partial w_0}{\partial \eta} \frac{\partial v_1}{\partial \xi} - \frac{u_r}{\eta} \frac{\partial w_1}{\partial \eta} \frac{\partial v_0}{\partial \xi} - \frac{1}{\eta} \frac{\partial w_0}{\partial \eta} \frac{\partial v_0}{\partial \eta} \\ & + \frac{u_r}{\eta} \pi \delta \frac{\partial v_1}{\partial \xi} \sin \pi \eta + \frac{1}{\eta} \pi \delta \frac{\partial v_0}{\partial \eta} \sin \pi \eta \end{aligned} \quad (1.65)$$

We substitute w_0, w_1, v_0, v_1 and ψ_0 into equation (1.65), and integrate the above equation, then we can obtain the solution of the boundary layer type for ψ_1 .

According to the above steps we can successively determine $w_n, v_n, \varphi_n, \psi_n$ ($n=0, 1, 2, \dots, N$), and substituting them into (1.9), we obtain the uniformly valid asymptotic solutions of N th-order for the deflection function W and stress function Φ , we confine ourselves to the approximate solutions of first-order, for this case we obtain the following:

$$\begin{aligned} W = & -\frac{q}{4a_3} \ln \left(\frac{1 - (1-\mu)r^2/(3-\mu)}{1 - (1-\mu)/(3-\mu)} \right) - \delta (1 + \cos \pi r) \\ & - \delta^2 \left\{ \frac{q\pi^2 c}{16a_3} \ln \left(\frac{a_1 + a_3 r^2}{a_1 + a_3} \right) + \frac{qc}{16} \int_1^r \frac{(1 - \cos 2\pi r)}{r(a_1 + a_3 r^2)} dr \right. \\ & - \frac{qc}{8} \int_1^r \frac{\sin 2\pi r}{a_1 + a_3 r^2} dr - \frac{q}{2} \int_1^r \frac{(e_1 r + e_3 r^3 + e_6 r^5)}{(a_1 + a_3 r^2)^2} dr \left. \right\} \\ & + \varepsilon \left\{ \frac{b_3 q}{4a_3^2} \ln \left(\frac{a_1 + a_3 r^2}{a_1 + a_3} \right) - \frac{q}{4a_3} \left(\frac{b_1 + b_3 r^2}{a_1 + a_3 r^2} - \frac{b_1 + b_3}{a_1 + a_3} \right) - c_0 (1) \right. \\ & + \frac{q\delta^2}{16} \int_1^r \frac{(b_1 + b_3 r^2) [\pi^2 c r^2 - \pi c r \sin 2\pi r + (1 - \cos 2\pi r) c/2]}{r(a_1 + a_3 r^2)^3} dr \\ & + \frac{q\delta^2}{2} \int_1^r \frac{e_1 r^2 + e_3 r^4 + e_6 r^6}{(a_1 + a_3 r^2)} dr \left. \right\} \\ & + \varepsilon \left\{ \xi(r) c_0 (1) \exp \left[-\varepsilon^{-1} \int_r^1 [(a_1 + a_3 \eta^2) + \delta^2 (e_1 + e_3 \eta^2 + e_6 \eta^4)]^{1/2} d\eta \right. \right. \\ & + \int_r^1 A^{1/2} \left\{ a_1 + 4a_3 \eta^2 + \delta^2 (e_1 + 4e_3 \eta^2 + 7e_6 \eta^4) \right. \\ & + [(a_1 + a_3 \eta^2) + \delta^2 (e_1 + e_3 \eta^2 + e_6 \eta^4)]^{1/2} \\ & \cdot (b_1 \eta + b_3 \eta^3) \left. \right\} d\eta \left. \right\} + O(\varepsilon^2) \end{aligned} \quad (1.66)$$

$$\begin{aligned} \frac{d\Phi}{dr} = & \{ (a_1 r + a_3 r^3) + \delta^2 (e_1 r + e_3 r^3 + e_5 r^5) \} \\ & + \varepsilon (b_1 r + b_3 r^3) - \varepsilon^2 \xi(r) \left\{ \frac{1}{8r} c_0^2(r) \exp(-2\xi) \right. \\ & + \frac{1}{ru_r} \left\{ \left[\frac{qr}{2(a_1 + a_3 r^2)} + \frac{q\delta^2}{8(a_1 r + a_3 r^3)} (\pi^2 cr^2 \right. \right. \\ & \left. \left. - \pi cr \sin 2\pi r + \frac{c}{2}(1 - \cos 2\pi r) \right] \right. \\ & \left. \left. - \frac{qr\delta^2}{2(a_1 + a_3 r^2)} (e_1 r + e_3 r^3 + e_5 r^5) \right\} c_0(r) \exp(-\xi) \right\} + O(\varepsilon^2) \quad (1.67) \end{aligned}$$

II. Remarks

1. From equations (1.66) and (1.67), we can see that when the initial deflection vanishes, the solutions of a circular thin plate with initial deflection is reduced to the solutions of the problems of the nonlinear bending of a circular thin plate^[6].

$$\begin{aligned} W = & -\frac{q}{4a_3} \ln \left(\frac{1 - (1-\mu)r^2/(3-\mu)}{1 - (1-\mu)/(3-\mu)} \right) + \varepsilon \left\{ \frac{b_3 q}{4a_3^2} \ln \left(\frac{a_1 + a_3 r^2}{a_1 + a_3} \right) \right. \\ & \left. - \frac{q}{4a_3} \left(\frac{b_1 + b_3 r^2}{a_1 + a_3 r^2} - \frac{b_1 + b_3}{a_1 + a_3} \right) - c_0(1) \right\} \\ & + \varepsilon \eta(r) \left\{ \frac{q}{2(a_1 + a_3)^{3/2}} \exp \left[-\varepsilon^{-1} \int_r^1 (a_1 + a_3 \eta^2)^{1/2} d\eta \right. \right. \\ & \left. \left. + \int^1 \frac{[a_1 + 4a_3 \eta^2 + (a_1 + a_3 \eta^2)^{1/2} (b_1 \eta + b_3 \eta^3)]}{2\eta(a_1 + a_3 \eta^2)} d\eta \right] \right\} + O(\varepsilon^2) \quad (2.1) \\ \frac{d\Phi}{dr} = & (a_1 r + a_3 r^3) + \varepsilon (b_1 r + b_3 r^3) \\ & - \varepsilon^2 \xi(r) \left\{ \frac{c_0^2(\eta)}{8\eta} \exp(-2\xi) + \frac{qc_0(\eta)}{2u_r(a_1 + a_3 \eta^2)} \exp(-\xi) \right\} + O(\varepsilon^2) \quad (2.2) \end{aligned}$$

Hence, when the initial deflection is very small, the circular thin plate does not produce the jumping phenomenon.

2. If the initial deflection is largish, and the signs of the initial deflection with the intensity of the transverse load are opposite, when the intensity of the transverse load reach certain value, which should produce the jumping phenomenon. Deflection curves for the circular thin plate with initial deflection, as shown in Fig.1.

3. In this paper, the boundary value problems of the nonlinear differential equations which involve two small independent parameters are studied by means of the associated method of the singular perturbation with the normal perturbation. We call this method the method of mixed perturbation. The above method is also applied to the boundary problems involving many small independent parameters.

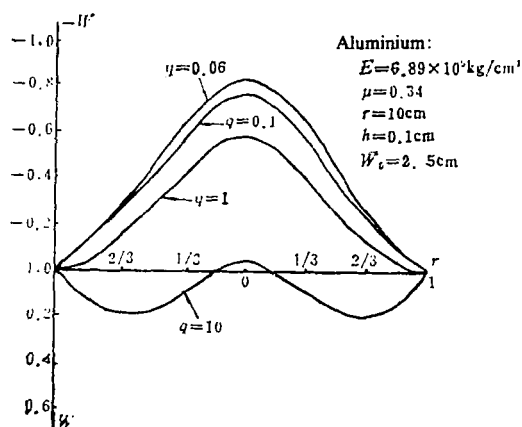


Fig.1. Deflection curves for a circular thin plate with initial deflection

References

- [1] Chien Wei-zang, Asymptotic behavior of a thin clamped circular plate under uniform pressure at very large deflection, *Sci. Rep. Nat. Tsinghua Univ. Ser.A*, 5(1948), 71-94.
- [2] Jiang Fu-ru, Unsymmetrical bending problems for the annular and circular thin plates under various supporting condition, *Applied Math. and Mech.*, 3,5(1982).
- [3] Jiang Fu-ru, On singular perturbations of elliptic equation, *Fudan Journal* Edition on Natural Science), 4,(1978), 29—37. (in Chinese)
- [4] Jiang Fu-ru, Some applications of perturbation method in thin plate bending problems. *Applied Math. and Mech.*, 1,1,(1980).
- [5] Qin Sheng-li, Zhang Ai-shu, On the problems of buckling of an annular thin plate, *Applied Math. and Mech.* 6,2,(1985).
- [6] Qin Sheng-li, Zhang Ai-shu, Kang Liang-cheng, On the problems of the nonlinear bending of a circular thin plate, *The proceedings of the First East China Conference on Solid Mechanics* (1984).
- [7] O'Malley, R.E., *Introduction to Singular Perturbations*, Academic Press(1974).
- [8] Wharmille, A.C., *Flexible Plates and Flexible Shells*, Science Press (1959), 162—193. (Chinese version)