

## REFLECTION AND RADIATION OF A WAVE SYSTEM AT THE OPEN END OF A SUBMERGED ELASTIC PIPE

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### Abstract

*The reflection and radiation of a wave system at the open end of a submerged semi-infinite elastic pipe are studied. This wave system consists of a flexural wave in the pipe, an acoustic surface wave in the fluid exterior to the pipe and an acoustic wave in the pipe's interior. Fourier transform techniques are used to formulate this semi-infinite geometry problem rigorously as a Wiener-Hopf type equation. An approximate solution is obtained by using a perturbation method in which the ratio of the mass densities of the fluid and the pipe material is regarded as a small parameter. The calculation of the reflection coefficient is emphasized, and the polar plots of the radiation coefficient are also presented.*

### I. Introduction and Formulation

Sound radiation from the open end of a pipe is one of the classical problems in acoustics. The reflection and radiation of axially symmetric sound waves, which propagate towards the open end of a rigid pipe of circular cross-section, has been analyzed in detail. First approximate solutions for  $a/\lambda \ll 1$ , where  $a$  is the pipe radius and  $\lambda$  is the wavelength, were presented by Helmholtz<sup>[1]</sup> and Rayleigh<sup>[2]</sup>. Rigorous and explicit expressions for the reflection coefficient and the radiated field were given by Levine and Schwinger<sup>[3]</sup> and Vainstein<sup>[4]</sup>. The scattering of a plane wave with arbitrary angle of incidence by a semi-infinite rigid rod (or pipe) was studied by Jones<sup>[5]</sup>, and we may regard [3] as a special case of [5] when the angle of incidence is  $\pi$ . Jones' work was extended by Williams<sup>[6]</sup> to the case in which the length of the pipe is finite.

To the author's knowledge, the topic of the present paper, i.e. the radiation from the open end of a submerged deformable pipe under the influence of flexural pipe waves, has not been analyzed before.

Consider a thin-walled open pipe submerged in a fluid with wall thickness  $h$ , mid-surface radius  $a$  and semi-infinite length. The geometry is shown in Fig. 1. It is assumed that  $h/a \ll 1$  and  $\lambda/a > 30$ , thus the Euler-Bernoulli beam theory are applicable<sup>[8]</sup> and we may write the beam equation as

$$EI \frac{\partial^4 w}{\partial x^4} + \rho_r A \frac{\partial^2 w}{\partial t^2} = p(x, t) \quad (1.1)$$

where  $w(x, t)$  is the transverse displacement (in the  $z$ -direction),  $EI$  is the bending stiffness ( $E$  = Young's modulus),  $\rho_r A$  is the mass per unit length ( $A$  = cross-sectional area) and

$$I = \pi a^3 h, \quad A = 2\pi a h \quad (1.2a, b)$$

$$p(x, t) = \int_0^{2\pi} (p_i - p_e)_{r=a} \cos \theta \cdot a d\theta \quad (1.3)$$

where  $p_i(r, \theta, x, t)$  and  $p_s(r, \theta, x, t)$  are the fluid pressure in the interior and the exterior of the pipe, respectively. The pressure and the particle velocity are given by

$$p = -\rho_f \dot{\phi}, \quad \vec{u} = \nabla \phi \quad (1.4a, b)$$

where  $\rho_f$  is the mass density of the fluid, and  $\phi(r, \theta, x, t)$  is the acoustic potential in the fluid governed by the differential equation

$$\nabla^2 \phi = \frac{1}{c_f^2} \ddot{\phi} \quad (1.5)$$

where  $c_f$  is the sound speed. The pipe interacts with the fluid via the pressure, and this fact is shown by the continuity of the radial component of the velocity of the particles:

$$\dot{w}(x, t) \cos \theta = \left. \frac{\partial \phi}{\partial r} \right|_{r=a} \quad (1.6)$$

According to paper [7] written by the author et al, an infinite submerged pipe can support a wave system of the form

$$\phi(r, \theta, x, t) = \Phi(r) \cos \theta \exp[i(kx - \omega t)] \quad (1.7)$$

$$w(x, t) = \frac{i}{\omega} B \exp[i(kx - \omega t)] \quad (1.8)$$

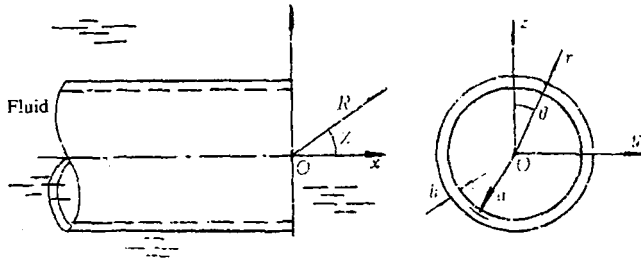


Fig. 1

where the factor  $i/\omega$  is introduced for convenience,  $B$  is a constant, and

$$\Phi(r) = \begin{cases} B J_1(qr)/qJ_1'(qa), & r \leq a \\ B H_1^{(1)}(qr)/qH_1^{(1)'}(qa), & r \geq a \end{cases} \quad (1.9a, b)$$

$$q^2 = k_f^2 - k^2, \quad k_f^2 = \omega^2/c_f^2 \quad (1.10a, b)$$

where  $J_1$ ,  $H_1^{(1)}$  are the Bessel function and the Hankel function of the first kind of order one, respectively. The wave number  $k$  and the frequency  $\omega$  in (1.7), (1.8) should satisfy the dispersion equation. After introducing the dimensionless frequency  $\Omega$  and other dimensionless quantities

$$\Omega = (\rho_f A a^4 / EI)^{1/2} \omega = (\sqrt{2} a / c_f) \omega, \quad c_p = (E / \rho_f)^{1/2} \quad (1.11)$$

$$\eta = ka, \quad \alpha^2 = c_p^2 / 2c_f^2, \quad \bar{q}^2 = \alpha^2 \Omega^2 - \eta^2, \quad \epsilon = \rho_f / \rho_p \quad (1.12a, b, c, d)$$

this dispersion equation may be written as

$$\bar{q}^2 J_1'(\bar{q}) H_1^{(1)'}(\bar{q}) (\eta^4 - \Omega^2) = 2i\epsilon \Omega^2 / (A/a^2) \quad (1.13)$$

According to [7], we know that when the frequency  $\Omega$  is smaller than a cut-off frequency given by  $\Omega_c = (1 + (2\pi\epsilon / (A/a^2))^{\frac{1}{2}}) / a^2$ , there is a surface wave (1.7) (relative to the pipe) propagating in the  $x$ -direction, but when  $\Omega > \Omega_c$  the corresponding wave in the fluid propagates in the  $r$ -direction as well as in the  $x$ -direction, because the dispersion equation has no real root and therefore  $\Phi(r)$  is also complex. For a steel pipe in water, density ratio  $\epsilon = 0.128$ , and we choose  $A/a^2 = 0.5$  (corresponding to  $h/a = 0.08$ ), then we have  $\Omega_c = 0.273$  and correspondingly,  $\eta_c = 0.664$ .

In the present paper, we consider the reflection of the incident wave system governed by equations (1.7)–(1.10) by the open end  $x = 0$  of a semi-infinite pipe and the accompanying radiation with the form of spherical wave into the fluid from the open end. In the rest of this paper, we will add the subscript "0" to represent the incident wave system, e.g., the incident wave in the fluid is  $\phi_0(r, \theta, x, t) = \Phi_0(r) \cos\theta \exp[i(k_0 x - \omega t)]$ , etc. The problem considered here differs from [3] in that the pipe wall is not rigid and the incident wave is not axially symmetric.

## II. Reducing to a Wiener-Hopf Equation

The total potential (total wave field) may be written as

$$\phi' = [\Phi_0(r) \exp[i k_0 x] + \phi(r, x)] \cos\theta \exp[-i\omega t] \quad (2.1)$$

where the unknown function  $\phi(r, x)$  represents the reflected and radiated fields. The wave equation (§ 5) implies that  $\phi$  must satisfy

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial x^2} + \left(k_0^2 - \frac{1}{r^2}\right) \phi = 0 \quad (2.2)$$

The corresponding transverse displacement of the pipe is

$$w(x) = \frac{i}{\omega} \phi'(a, x) \quad (2.3)$$

where the prime denotes differentiation with respect to  $r$ . In the range  $-\infty < x \leq 0$ ,  $w$  and  $\phi$  must satisfy eq. (1.1), so we have

$$EI \frac{\partial^4}{\partial x^4} \phi'(a, x) - \rho_s A \omega^2 \phi'(a, x) = -\rho_0 \omega^2 [\phi(a+0, x) - \phi(a-0, x)] \quad (-\infty < x \leq 0) \quad (2.4)$$

We note that  $\phi'(r, x)$  (the particle velocity) is continuous across the surface  $r = a$ , while  $\phi(r, x)$  (the pressure) may be discontinuous for  $x < 0$ . The bending moment and transverse shear force due to fluid pressure on the thin-walled cross-section are neglected, and hence the boundary conditions are

$$\left. \begin{aligned} \left[ \frac{\partial^2}{\partial x^2} \phi'_0(a, x) + \frac{\partial^2}{\partial x^2} \phi'(a, x) \right]_{x=0} &= 0 \\ \left[ \frac{\partial^3}{\partial x^3} \phi'_0(a, x) + \frac{\partial^3}{\partial x^3} \phi'(a, x) \right]_{x=0} &= 0 \end{aligned} \right\} \quad (2.5)$$

where

$$\phi'_0(a, x) = \Phi'_0(a) \exp[i k_0 x] = B \exp[i k_0 x] \quad (2.6)$$

In addition we require  $\phi(r, x)$  to satisfy the radiation condition

$$R \left( \frac{d\phi}{dR} - i k_r \phi \right) \rightarrow 0 \quad \text{as } R \rightarrow \infty \quad (2.7)$$

where  $R \equiv (r^2 + x^2)^{\frac{1}{2}}$ , thus we can determine the solution of  $\phi(r, x)$  uniquely.

We use the Fourier transform techniques to solve this problem and define the Fourier transform of  $\phi(r, x)$  as

$$\Phi(r, k) = \int_{-\infty}^{\infty} \exp[ikx] \phi(r, x) dx \quad (2.8)$$

It follows from (2.2) that  $\Phi(r, k)$  is given by

$$\Phi(r, k) = \begin{cases} A_1(k) H_1^{(1)}(qr), & r \geq a \\ A_2(k) J_1(qr), & r \leq a \end{cases} \quad (2.9)$$

where  $q$  is defined by (1.10a), and  $A_1(k)$ ,  $A_2(k)$  are to be determined. The continuity of  $\phi'(r, x)$  at  $r = a$  gives the following relation between  $A_1(k)$  and  $A_2(k)$

$$A_1(k) H_1^{(1)'}(qa) = A_2(k) J_1'(qa) \quad (2.10)$$

We define

$$\Phi_- = \int_{-\infty}^0 \exp[ikx] \phi(r, x) dx, \quad \Phi_+ = \int_0^{\infty} \exp[ikx] \phi(r, x) dx \quad (2.11a, b)$$

hence

$$\Phi(r, k) = \Phi_-(r, k) + \Phi_+(r, k) \quad (2.12)$$

where  $\Phi_+$  and  $\Phi_-$  are analytic in the upper and lower halves of the complex  $k$ -plane, respectively. Next, we multiply (2.4) by  $\exp[ikx]$ , and integrate from  $x = -\infty$  to  $x = 0$ . By using the method of integration by parts, we have

$$\begin{aligned} EIk^4 \Phi_-(a, k) + ik^3 EI \phi'(a, 0) - k^2 EI \frac{\partial}{\partial x} \phi'(a, 0) - ikEI \frac{\partial^2}{\partial x^2} \phi'(a, 0) \\ + EI \frac{\partial^3}{\partial x^3} \phi'(a, 0) - \rho_f A \omega^2 \Phi_-(a, k) = -\rho_f \pi a \omega^2 V_-(a, k) \end{aligned} \quad (2.13)$$

where

$$V_-(a, k) \equiv \Phi_-(a+0, k) - \Phi_-(a-0, k) \quad (2.14)$$

The boundary conditions (2.5) (2.6) and the continuity of  $\phi'$  yield

$$\Phi_+(a+0, k) - \Phi_+(a-0, k) = \frac{-2B}{\pi q_0^2 a J_1'(q_0 a) H_1^{(1)'}(q_0 a)} \cdot \frac{1}{k + k_0} \quad (2.15)$$

where  $q_0$  is defined as

$$q_0^2 = k_i^2 - k_0^2$$

and  $k_0$  is the wave number of the incident wave. By using (2.9), (2.10) and (2.15), we obtain

$$V_-(a, k) = \frac{-2iA_1}{\pi q a J_1'(qa)} + \frac{2B}{\pi q_0^2 a J_1'(q_0 a) H_1^{(1)'}(q_0 a)} \cdot \frac{1}{k + k_0} \quad (2.16)$$

Eliminating the function  $A_1(k)$  of (2.16) and substitution  $V_-(a, k)$  into (2.13) yield the following equation for  $\Phi_+'(a, k)$  and  $\Phi_-'(a, k)$

$$[q^2 J_1'(qa) H_1^{(1)'}(qa) (EIk^4 - \rho_f A \omega^2) - 2i\rho_f \omega^2] \Phi_-'(a, k) = 2i\rho_f \omega^2 \Phi_+'(a, k)$$

$$\begin{aligned}
& -2\rho_f \omega^2 \frac{q^2 J_1'(qa) H_1^{(1)'}(qa) B}{q_0^2 J_1'(q_0 a) H_1^{(1)'}(q_0 a) (k+k_0)} \\
& -BEI q^2 J_1'(qa) H_1^{(1)'}(qa) [ik^3 C_1 - k^2 k_0 C_2 - i k k_0^2 + i k_0^3] \quad (2.17)
\end{aligned}$$

where the unknown constants  $C_1$  and  $C_2$  have been introduced:

$$C_1 = \phi'(a, 0)/B \quad (2.18)$$

$$C_2 = \frac{\partial}{\partial x} \phi'(a, 0)/B k_0 \quad (2.19)$$

They will be determined in the course of solving the problem.

In dimensionless form, (2.17) may be rewritten as

$$\begin{aligned}
& [L(\eta)(\eta^4 - \Omega^2) + \epsilon(a/h)\Omega^2] \Psi_-(\eta) = -\epsilon(a/h)\Omega^2 \Psi_+(\eta) \\
& - \frac{i\epsilon(a/h)\Omega^2}{\eta + \xi} \cdot \frac{L(\eta)}{L(\xi)} - L(\eta)(i\eta^3 C_1 - \eta^2 \xi C_2 - i\eta \xi^2 + i\xi^3) \quad (2.20)
\end{aligned}$$

where we have used that  $A = 2\pi a h$ , and introduced new dimensionless quantities

$$\xi = k_0 a \quad (2.21)$$

$$L(\eta) = i\pi \bar{q}^2 J_1'(\bar{q}) H_1^{(1)'}(\bar{q}) \quad (2.22)$$

$$\Psi_{\pm}(\eta) = \Phi_{\pm}'(a, k)/Ba \quad (2.23)$$

The functions  $\Psi_+(\eta)$  and  $\Psi_-(\eta)$  in (2.20) are analytic in the upper and lower halves of the complex  $\eta$ -plane, respectively. They have a common region of analyticity in the neighborhood of the real axis. Therefore, (2.20) is a Wiener-Hopf equation from which, in principle, both  $\Psi_+(\eta)$  and  $\Psi_-(\eta)$  can be determined.

### III. Perturbation Solution to the Wiener-Hopf Equation (2.20)

To solve the Wiener-Hopf equation (2.20) in the usual manner requires the factorization of the term multiplying  $\Psi_-(\eta)$  (i.e. the kernel function) into terms that are analytic in the upper and lower halves of the complex  $\eta$ -plane. It appears, however, that this kernel is too complex to factorize, except in the formal integral form.

Noting that the density ratio  $\epsilon$  is generally a small number (e.g., for a steel pipe in water,  $\epsilon = 0.128$ ), we expand the unknown functions  $\Psi_+$  and  $\Psi_-$  and the unknown constants  $C_1$  and  $C_2$  in the terms of asymptotic series:

$$\Psi_{\pm} = \Psi_{\pm}^{(0)} + \epsilon \Psi_{\pm}^{(1)} + \epsilon^2 \Psi_{\pm}^{(2)} + \dots \quad (3.1)$$

$$C_j = C_j^{(0)} + \epsilon C_j^{(1)} + \epsilon^2 C_j^{(2)} + \dots \quad (j=1, 2) \quad (3.2)$$

In addition, we use the dispersion equation (1.13) for  $\eta(\Omega)$  to rewrite the term  $(\eta^4 - \Omega^2)$  in (2.20) as

$$\eta^4 - \Omega^2 = (\eta^4 - \xi^4) - \epsilon \frac{a}{h} \frac{\Omega^2}{L(\xi)} \quad (3.3)$$

Substituting (3.1)–(3.3) into (2.20) and equating the coefficients of each power of  $\epsilon$  on both sides of the equation yields a sequence of Wiener-Hopf equations.

#### 3.1 The zero-order solution

Equating the coefficients of  $\epsilon^0$  gives

$$L(\eta)(\eta^4 - \xi^4)\Psi_{-}^{(0)}(\eta) = -L(\eta)(i\eta^3 C_1^{(0)} - \eta^2 \xi C_2^{(0)} - i\eta \xi^2 + i\xi^3) \quad (3.4)$$

This Wiener-Hopf equation is a degenerate one because it does not involve  $\Psi_{+}^{(0)}(\eta)$ . Eliminating the terms  $L(\eta)$  on both sides and requiring that  $\Psi_{-}^{(0)}(\eta)$  be analytic at  $\eta = -\xi$  and  $\eta = -i\xi$ , we can determine constants  $C_1^{(0)}$  and  $C_2^{(0)}$ , and obtain

$$\Psi_{-}^{(0)}(\eta) = \frac{1}{\eta - \xi} + \frac{1-i}{\eta - i\xi} \quad (3.5)$$

then we find

$$\phi'^{(0)}(a, k) = \left[ \frac{B}{k - k_0} + \frac{B(1-i)}{k - ik_0} \right] + \Phi_{+}^{(0)}(a, k)$$

By taking the Fourier inversion, we have

$$\phi'^{(0)}(a, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-ikx] \left\{ \left[ \frac{B}{k - k_0} + \frac{B(1-i)}{k - ik_0} \right] + \Phi_{+}^{(0)}(a, k) \right\} dk$$

We obtain from the calculus of residues that

$$\phi'^{(0)}(a, x) = iB \exp[-ik_0 x] + (1+i)B \exp[k_0 x] \quad (x \leq 0)$$

By comparison with (2.6), we conclude that the reflection coefficient is  $i$ , and the coefficient of end resonance is  $1+i$ . This result is in agreement with the well known conclusion with fluid absent (e.g., see [9], noting that  $\phi'(a, x)$  corresponds to the transverse displacement of the beam). Besides, it affirms that rewriting (3.3) is necessary to obtain the correct result.

### 3.2 The first-order solution

The first-order equation is

$$\begin{aligned} sL(\eta)(\eta^4 - \xi^4)\Psi_{-}^{(1)}(\eta) &= \Psi_{+}^{(0)}(\eta) + \Psi_{-}^{(0)}(\eta)[1 - L(\eta)/L(\xi)] \\ &+ \frac{iL(\eta)}{L(\xi)(\eta + \xi)} - sL(\eta)(i\eta^3 C_1^{(1)} - \eta^2 \xi C_2^{(1)}) \end{aligned} \quad (3.6)$$

where the constant

$$s \equiv -[2\pi\Omega^2 a^2/A]^{-1} = -(h/a)/\Omega^2 \quad (3.7)$$

This is a true Wiener-Hopf equation, unlike that of (3.4), in which there are two unknown functions,  $\Psi_{-}^{(1)}$  and  $\Psi_{+}^{(0)}$  to be determined, and a known function  $\Psi_{-}^{(0)}$  given by (3.5). Although the kernel  $L(\eta)(\eta^4 - \xi^4)$  in (3.6) remains complicated, it has been substantively simplified from the kernel in (2.20). from here we can see how the perturbation method plays a key role in this problem.

We solve eq.(3.6) by first factorizing  $L(\eta)$  as

$$L(\eta) = L^{-}(\eta)/L^{+}(\eta) \quad (3.8)$$

where  $L^{-}$  and  $L^{+}$  are analytic functions in the lower and upper half-planes, respectively. The function  $L(\eta)$  is similar to the function in [3], but more complicated than that one. In Appendix, the factorization of  $L(\eta)$  is briefly described.

We multiply eq. (3.6) by  $L^{+}(\eta)$ , then add and subtract certain terms to obtain

$$sL^{-}(\eta)(\eta^4 - \xi^4)\Psi_{-}^{(1)}(\eta) + sL^{-}(\eta)[i\eta^3 C_1^{(1)} - \eta^2 \xi C_2^{(1)}] + \frac{L^{-}(\eta)}{L(\xi)}\Psi_{-}^{(0)}(\eta)$$

$$\begin{aligned} & -\frac{i[L^-(\eta)-L^-(-\xi)]}{L(\xi)(\eta+\xi)}-\frac{L^+(\xi)}{\eta-\xi}-\frac{1-i}{\eta-i\xi}L^+(i\xi)=L^+(\eta)\Psi_+^{(0)}(\eta) \\ & +\frac{iL^-(-\xi)}{L(\xi)(\eta+\xi)}+\frac{L^+(\eta)-L^+(\xi)}{\eta-\xi}+\frac{1-i}{\eta-i\xi}[L^+(\eta)-L^+(i\xi)] \end{aligned} \quad (3.9)$$

The left- and right-hand sides of this equation are analytic in the lower and upper halves of the  $\eta$  plane, respectively. By the usual arguments it may be shown first that each side of the equation is equal to an entire function, and secondly that the entire function is identically zero. These conclusions imply that

$$\Psi_+^{(0)}(\eta)=\left[-\frac{iL^-(-\xi)}{L(\xi)(\eta+\xi)}+\frac{L^+(\xi)}{\eta-\xi}+\frac{1-i}{\eta-i\xi}L^+(i\xi)\right]/L^+(\eta)-\Psi_-^{(0)}(\eta) \quad (3.10)$$

$$\begin{aligned} \Psi_-^{(1)}(\eta)= & \frac{\eta^2\xi C_2^{(1)}-i\eta^3 C_1^{(1)}}{\eta^4-\xi^4}+\left\{\frac{i[L^-(\eta)-L^-(-\xi)]}{L(\xi)(\eta+\xi)}+\frac{L^+(\xi)}{\eta-\xi}+\frac{1-i}{\eta-i\xi}L^+(i\xi)\right. \\ & \left.-\frac{L^-(\eta)}{L(\xi)}\Psi_-^{(0)}(\eta)\right\}/sL^-(\eta)(\eta^4-\xi^4) \end{aligned} \quad (3.11)$$

where the constants  $C_1^{(1)}$  and  $C_2^{(1)}$  are determined by the requirement that  $\Psi_-^{(1)}(\eta)$  be analytic at  $\eta=-\xi$  and  $\eta=-i\xi$ .

The author has also considered the second-order solution, which is lengthy and troublesome. The result obtained under certain assumptions is shown in Fig. 2, Section VI.

#### IV. The Reflected Wave on the Pipe

After determining  $\Psi_+(\eta)$  and  $\Psi_-(\eta)$ , the transverse displacement of the pipe due to the reflected wave may be calculated. For any  $x$ , by taking the Fourier inversion, it follows

$$\phi'(a, x)=\frac{B}{2\pi}\int_{-\infty}^{\infty}[\Psi_+(\eta)+\Psi_-(\eta)]\exp[-i\eta x/a]d\eta \quad (4.1)$$

For  $x < 0$ , the integral involving  $\Psi_+(\eta)$  is zero and we have

$$w(x)=\frac{i}{\omega}\phi'(a, x)=-\frac{iB}{2\pi\omega}\int_{-\infty}^{\infty}\Psi_-(\eta)\exp[-i\eta x/a]d\eta \quad (4.2)$$

We obtain the zero-order perturbation from Section III for  $x < 0$

$$w(x)\sim\frac{iB}{\omega}[i\exp[-ik_0x]+(1+i)\exp[k_0x]] \quad (4.3)$$

where the second term on the right-hand side is the term of end resonance. As  $x \rightarrow -\infty$  this term disappears and the displacement is just that of the reflected wave.

In general, the asymptotic displacement as  $x \rightarrow -\infty$  is

$$w(x)\sim\frac{iB}{\omega}\mathcal{R}(\Omega)\exp[-ik_0x] \quad (4.4)$$

where the reflection coefficient  $\mathcal{R}(\Omega)$  is the contribution from the pole at  $\eta=\xi$ , i.e.,

$$\mathcal{R}(\Omega)=i\operatorname{Res}[\Psi_-(\eta)]_{\eta=\xi} \quad (4.5)$$

Expanding  $\mathcal{R}(\Omega)$  in powers of  $\epsilon$ , we have

$$\mathcal{R}(\Omega)=i+\epsilon R_1+\epsilon^2 R_2+\cdots \quad (4.6)$$

where the coefficient  $R_j$  is determined from  $\Psi_{-}^{(j)}(\eta)$ ,  $j=1,2,\dots$ . From (3.11), we have calculated the first term  $R_1$  as

$$R_1 = \frac{1}{8s\xi^4} \left\{ 4i L^+(\xi) L^+(i\xi) + 2[L^+(i\xi)]^2 - [L^+(\xi)]^2 - 4i \frac{L^+(i\xi)}{L^-(\xi)} - [L^-(\xi)]^{-2} + \frac{2i}{L(\xi)} \left[ \frac{M(\xi)}{L^-(\xi)} - M(-\xi) L^+(\xi) \right] \right\} \quad (4.7)$$

where

$$M(x) = x \frac{d}{dx} L^-(x) \quad (4.8)$$

## V. The Radiated Field in the Fluid

By taking the Fourier inversion, the radiated field in the fluid may be written as

$$\phi(r, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(r, k) \exp[-ikx] dk \quad (5.1)$$

where  $\Phi(r, k)$  can be obtained as

$$\Phi(r, k) = Ba[\Psi_+(ka) + \Psi_-(ka)] \begin{cases} H_1^{(1)}(qr)/qH_1^{(1)'}(qa), & r \geq a \\ J_1(qr)/qJ_1'(qa), & r \leq a \end{cases} \quad (5.2)$$

We consider the region  $r > a$  and introduce the coordinates  $R$  and  $\chi$  defined by (see Fig.1)

$$x = R \cos \chi, \quad r = R \sin \chi \quad (5.3)$$

We also assume that  $qr$  is large so that  $H_1^{(1)}(qr)$  may be approximated by its asymptotic expression

$$H_1^{(1)}(qr) \sim \left( \frac{2}{\pi qr} \right)^{\frac{1}{2}} \exp[i(qr - 3\pi/4)] \quad (5.4)$$

The radiated field is from (5.1):

$$\phi \sim \frac{Ba}{2\pi} \left( \frac{2}{k_f R \pi \sin \chi} \right)^{\frac{1}{2}} \exp[i\pi/4] \int_C \frac{\Psi(-k_f a \cos \gamma) \exp[ik_f R \cos(\gamma - \chi)]}{(\sin \gamma)^{1/2} H_1^{(1)'}(k_f a \sin \gamma)} d\gamma \quad (5.5)$$

where the substitution  $k = -k_f \cos \gamma$  has shifted the integration onto the contour  $C$  running from  $\pi - i\infty$  to  $\pi$ , then along the real axis to  $O$  and finally to  $i\infty$ . Also, in (5.5),  $\Psi = \Psi_+ + \Psi_-$ .

Substitution of  $\Psi_+^{(0)}$  from (3.10) and  $\Psi_-^{(0)}$  from (3.5) into (5.5) yields the first approximation of radiated field. We see from eq. (3.10) that it has complicated analytic properties. As  $R \rightarrow \infty$  we can split the integral in eq. (5.5) into separate contributions from poles, branch cuts and stationary phase points. By the method of stationary phase, the latter contribution is

$$\phi_{sp}(R, \chi) = D(\chi) \exp[ik_f R] / k_f R \quad (5.6)$$

where the radiation coefficient  $D(\chi)$  is

$$D(\chi) = \frac{Ba}{\pi} \frac{\Psi^{(0)}(-k_f a \cos \chi)}{\sin \chi H_1^{(1)'}(k_f a \sin \chi)} \quad (5.7)$$

The pole of  $\Psi^{(0)}(\eta)$  at  $\eta = \xi$  corresponds to the reflected wave on the pipe which propagates in the direction of  $x = -\infty$ . This pole is crossed by the steepest descent contour if



$\pi - \chi_0 < \chi < \pi$ , where  $\chi_0 = \arccos \left\{ \min \left[ \operatorname{Real} \left( \frac{k_0}{k_f}, \frac{k_f}{k_0} \right) \right] \right\}$ . The pole at  $\eta = -\xi$  is crossed if  $0 < \chi < \chi_0$ , its contribution exactly cancels the incident wave in the forward  $x$ -direction where there is no pipe to support such a wave. Finally, the pole at  $\eta = i\xi$  gives rise to an evanescent wave which decays rapidly away from the end of the pipe.

Thus, as  $R \rightarrow \infty$  the main contribution to the radiated field is the field  $\phi_s$ , of eq. (5.6). Its angular variation is determined by the radiation coefficient  $D(\chi)$  of eq. (5.7), where  $\Psi^{(0)}(\eta)$  is given by eq. (3.10) and (3.5).

## VI. Energy Analysis and Numerical Results

It is not difficult to compute the time-averaged energy transmission (over a period  $T = 2\pi/\omega$ ) across a cross-section of constant  $x$  as

$$\langle P^{\text{in}} \rangle = \langle P_f^{\text{in}} \rangle + \langle P_f^{\text{in}} \rangle \quad (6.1)$$

where the time-averaged energy transmission of the incident wave on the pipe is given by

$$\langle P_f^{\text{in}} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi} [\operatorname{Re}(-EIw'')\operatorname{Re}(\dot{w}') + \operatorname{Re}(EIw'')\operatorname{Re}(\dot{w})] dt = EIB^2 \frac{k_0^3}{\omega} \quad (6.2)$$

and the corresponding time-averaged energy transmission in the water for  $\Omega < \Omega_0$  is

$$\langle P_f^{\text{in}} \rangle = \langle P_f^{\text{in}} \rangle \frac{\epsilon \Omega^2}{q_0^2 \xi^2} \frac{a}{4h} \left\{ \frac{\int_0^1 [I_1(q_0 a v)]^2 v dv}{[I_1'(q_0 a)]^2} + \frac{\int_1^\infty [K_1(q_0 a v)]^2 v dv}{[K_1'(q_0 a)]^2} \right\} \quad (6.3)$$

where  $I_1$ ,  $K_1$  are the modified Bessel functions of the first and second kind, respectively.

An expression for the time-averaged energy transmission across a distant spherical surface may be written as

$$\langle P^R \rangle = \langle P_f^{\text{in}} \rangle \frac{\epsilon \Omega^2}{4\xi^2 k_f a (Ba)^2} \cdot \frac{a}{h} \int_0^\pi |D(\chi)|^2 \sin \chi d\chi \quad (6.4)$$

where  $D(\chi)$  is defined by (5.7).

As a valuable check, an alternative computation of the reflection coefficient can be based on energy balance

$$\langle P^{\text{in}} \rangle = \langle P^{\text{re}} \rangle + \langle P^R \rangle \quad (6.5)$$

because the reflected energy may be written as

$$\langle P^{\text{re}} \rangle = |\mathcal{R}(\Omega)|^2 \langle P^{\text{in}} \rangle \quad (6.6)$$

Substitution of (6.6) into (6.5) yields

$$|\mathcal{R}(\Omega)| = \left[ \frac{\langle P^{\text{in}} \rangle - \langle P^R \rangle}{\langle P^{\text{in}} \rangle} \right]^{1/2} \quad (6.7)$$

For the steel pipe in water in Section I, the absolute value of the reflection coefficient  $\mathcal{R}(\Omega)$  has been plotted versus the dimensionless frequency in Fig. 2. The zeroth approximation is  $|\mathcal{R}(\Omega)| = 1$ . For the first order approximation,  $|\mathcal{R}(\Omega)| = |i + \epsilon R_1|$  and the second

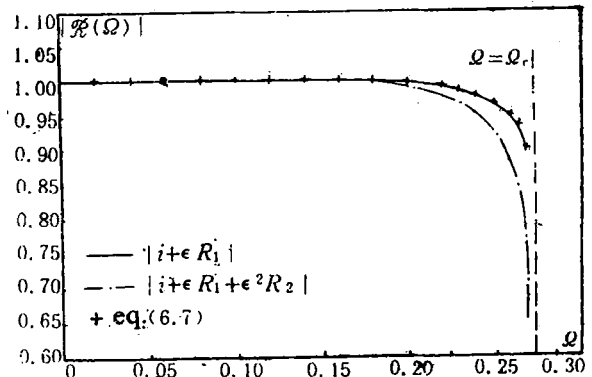


Fig. 2 The curves of radiation coefficient

order,  $|\mathcal{R}(\Omega)| = |i + \epsilon R_1 + \epsilon^2 R_2|$ . The expression given by (6.7) has also been plotted in Fig.2. Noting that (6.7) is of a kind of first order approximation, it is not surprised that (6.7) and  $|i + \epsilon R_1|$  are very close to each other. These curves in Fig.2 indicate that the amount of reflected energy decreases rapidly, and more energy is radiated into the fluid as the frequency of the incident wave comes close to the cut-off frequency.

Fig. 3 shows polar plots of  $|D(\chi)|$ , i.e. the radiation pattern of the far-field, for various values of the dimensionless frequency, where we choose  $Ba/\pi = 1$ . At low frequencies the radiation is strongest in the range  $\pi/4 < \chi < 3\pi/4$ . As the frequency approaches the cut-off frequency, the maximum values of  $|D(\chi)|$  occur at smaller and smaller angles  $\chi$ , until at  $\Omega = 0.27$ , the maximum values of  $|D(\chi)|$  are obtained in direction almost parallel to the pipe. It is also noted that the magnitudes of  $|D(\chi)|$  increase dramatically as  $\Omega$  increases.

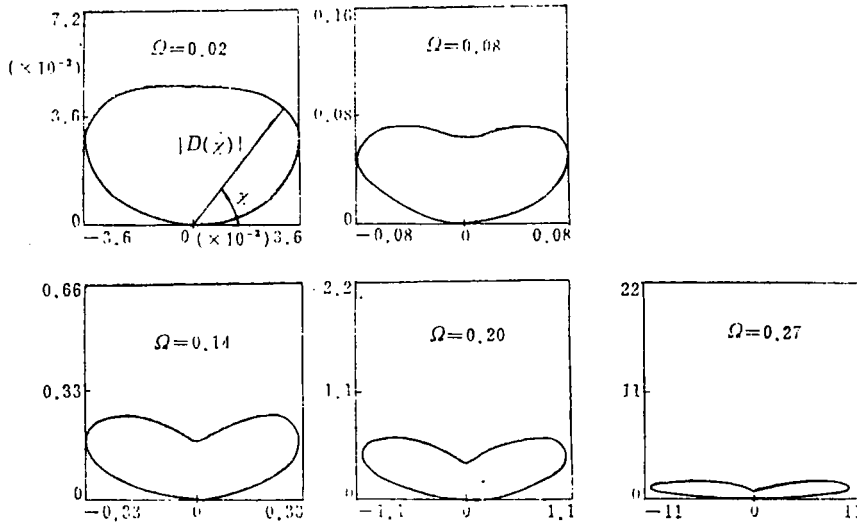


Fig.3 The radiation pattern of the far-field for various values of the frequency

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### Appendix: Factorization of $L(\eta)$ in (3.8)

The function is actually factorized in this work is  $L(\xi) = -\pi i(k_f^2 - \xi^2)a^2 H_1^{(1)'}[(k_f^2 - \xi^2)^{\frac{1}{2}}a] J_1'[(k_f^2 - \xi^2)^{\frac{1}{2}}a]$ , which differs from (2.22) only in sign. We outline the procedure of factorization as follows:

1. Apply Cauchy's integral formula to the function  $\log L(\xi)$  in a narrow rectangular domain (with sides parallel to the coordinate axes) and obtain the integral expressions of  $L^+$  and  $L^-$ , and the relation to them:  $L^-(\xi) = 1/L^+(-\xi)$ . This is the same as the usual way to factorize a function formally.

2. Derive the explicit analytical expression of the function  $f(\xi) = (d/d\xi) \log L^+(\xi)$ . In this paper,

$$f(\xi) = \frac{-1}{2\pi i} \int_{C_+} \frac{\log \{-\pi i(k_f^2 - t^2) H_1^{(1)'}[(k_f^2 - t^2)^{\frac{1}{2}}a] J_1'[(k_f^2 - t^2)^{\frac{1}{2}}a]\}}{(t - \xi)^2} dt \quad (1)$$

In the process of the derivation, the contour  $C_+$  shown in fig. 4 (a) is replaced by the contours  $\Gamma_1$ ,

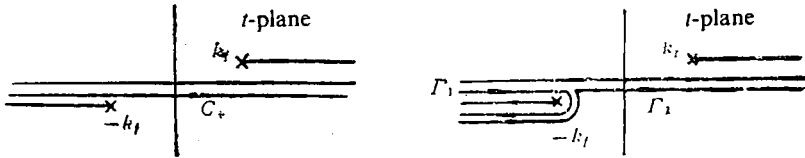


Fig. 4

$\Gamma_1$  shown in Fig. 4 (b). The integral along  $\Gamma_1$  is expressed in terms of integrals along the remainder of a closed contour in the lower half plane, and this closed contour runs in such a way that it does not include any zero of the function  $J_1'$  and  $H_1^{(1)'}'$  inside.

3. Integrate  $f(\xi)$  with respect to  $\xi$ , then take the inverse logarithm and determine the constant that appears in the integration. We can finally obtain the explicit expression of  $L^+(\xi)$ .

There is no zero of the function  $H_1^{(1)}[(k^2 - t^2)^{1/2}a]$  in paper [3] in the lower half of the  $t$ -plane, because  $H_1^{(1)}(z)$  has no zero in the lower right quadrant of the  $z$ -plane (the author computes the zero of  $H_1^{(1)}(z)$  as  $z_1^* = -0.4344 - 0.5595i$ ). However, here  $H_1^{(1)'}(z)$  does have a zero in the lower right quadrant (which the author computes as  $z_2^* = 0.5012 - 0.6435i$ ), it corresponds to a zero of  $H_1^{(1)'}[(k^2 - t^2)^{1/2}a]$  in the lower left quadrant of the  $t$ -plane at  $t_*$ , so that we have to avoid crossing the branch cut which is drawn from the point  $t = t_*$  and we finally have an extra term with the form  $1/(\xi - t_*)$  in the analytical expression of  $f(\xi)$ . By the way, we point out that the discussions of zeros of  $H_1^{(1)}(z)$  and  $H_1^{(1)'}(z)$  in the complex  $z$ -plane are not sufficient in the literature and with errors. For example, the conclusion " $H_1^{(1)}(z)$  has no zero for which  $-\pi/2 \leq \arg z \leq 3\pi/2$ " in [3] is not correct.

Finally, noting that  $f(\xi) = (dL^+(\xi)/d\xi)/L^+(\xi)$ , the derivate of  $L^-(\xi)$  may be written in terms of  $f$  and  $L^+$  as

$$\frac{dL^-(\xi)}{d\xi} = \frac{d}{d\xi} \left[ \frac{1}{L^+(-\xi)} \right] = \frac{-dL^+(-\xi)/d\xi}{[L^+(-\xi)]^2} = f(-\xi)/L^+(-\xi) \quad (2)$$

## References

- [1] Helmholtz, H., Theorie der Luftschwingungen in Röhren mit offenen Enden, *Crelle's J. Reine Angewandete Math.*, **57** (1860), 1 - 72.
- [2] Rayleigh, Lord, *Theory of Sound*, Vol. **2**, Dover (1945), 196 - 201, 487 - 491.
- [3] Levine, H. and J. Schwinger, On the radiation of sound from an unflanged circular pipe, *Phys. Rev.*, **73** (1948), 383 - 406.
- [4] Vainstein, L., The theory of sound waves in open tubes, *Zh. Tech. Fiz.*, **19** (1949), 911 - 930.
- [5] Jones, D.S., The scattering of a scalar wave by a semi-infinite rod of circular cross section, *Phil. Trans. Roy. Soc.*, **A247** (1955), 499 - 528.
- [6] Williams, W.E., Diffraction by a cylinder of finite length, *Proc. Camb. Phil. Soc.*, **52** (1956), 322 - 335.
- [7] Song, J., A.N. Norris and J.D. Achenbach, Acoustic radiation generated by local excitation of submerged beams and strings, *J. Sound Vib.*, **100** (1985), 108 - 121.
- [8] Kornecki, A., A note on beam-type vibrations of circular cylindrical shells, *J. Sound Vib.*, **14** (1971), 1 - 6.
- [9] Graff, K.F., *Wave Motion in Elastic Solids*, Ohio State Univ. Press (1975), 154 - 155.