Applied Mathematics and Mechanics
(English Edition, Vol.8, No.3, Mar. 1987)

Published by SUT,
Shanghai, China

# ON THE LINKING UP BETWEEN BINGHAM FLUID AND PLUGGED FLOW

Tsai Shu-tang (蔡树棠)

(*Department of Modern Mechanics, China Univerity of Science and Technology, Hefei; Shanghai Institute of Applied Mathematics and Mechanics, Shanghai*)

Jiang Yi-an (蒋贻安)

(*Department of Physics, Hangzhou Normal College, Hangzhou*)

(Received Dec. 4, 1985)

## Abstract

*When Bingham fluid is in motion, plugged flow often occurs at places far from the boundary walls. As there is not a decisive formula of constitutive relation for plugged flow, in some problems the solutions obtained may be indefinite. In this paper, annular flow and pipe flow are discussed, and unique solution is obtained in each case by utilizing the analytic property of shear stress. The solutions are identical in form with the commonly used formula for the pressure drop of mud flow in petroleum engineering.*

## I. Introduction

Mud is a necessary medium in petroleum engineering. Sometimes such mud can be treated as Bingham fluid. When Bingham fluid is in motion, plugged flow often occurs at places far from the boundary walls. The problem of linking-up between Bingham fluid and plugged flow arises then. As there is no decisive formula of constitutive relation for plugged flow, the solution may be indefinite. In this paper, unique solution is obtained for the case of annular flow and of pipe flow by utilizing the condition that at the interface between Bingham fluid and plugged flow the two stresses should be equal and the two velocities of flow should be equal. From the unique solution, the formula of the pressure drop in mud flow can be found under the approximate condition that the yielding stress and the annular radii difference are small. The formula obtained is identical in form with the commonly used formula in petroleum engineering.

## II. Several Kinds of the Problem of Linking up between Bingham Fluid and Plugged Flow

### (a) Annular flow

The equation of motion in this case is

$$-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})=0 \tag{2.1}$$

where $p$ is the pressure, $\sigma_{rz}$ is the shear stress, and $(r,\theta,z)$ are cylindrical coordinates. After integration we have

$$\sigma_{rz}=\frac{1}{2}\frac{\partial p}{\partial z}r+\frac{c}{r} \tag{2.2}$$

Where $c$ is the constant of integration.

For Bingham fluid, let $\tau_0$ be the yielding stress, $\eta$ be the rigidity modulus, $v_z$ be the axial velocity, then

$$\sigma_{rz}=\left(\frac{\tau_0}{|dv_z/dr|}+\eta\right)\frac{dv_z}{dr} \tag{2.3}$$

At the inner cylindrical boundary surface,

$$r=R_1,\quad v_z=0$$

At the outer cylindrical boundary surface,

$$r=R_2,\quad v_z=0$$

At the inner linking point of plugged flow and Bingham fluid,

$$r=R_1^0,\quad v_z=v_z^0,\quad dv_z/dr=0,\quad \sigma_{rz}^P=\sigma_{rz}^B$$

At the outer linking point of plugged flow and Bingham fluid,

$$r=R_2^0,\quad v_z=v_z^0,\quad \frac{dv_z}{dr}=0,\quad \sigma_{rz}^P=\sigma_{rz}^B$$

Where $v_z^0$ is the velocity of plugged flow, $\sigma_{rz}^P$ is the shear stress of plugged flow at the interface, $\sigma_{rz}^B$ is the shear stress of Bingham fluid at the interface. As the pressure gradient in Bingham fluid is the same as that in plugged flow, and the radii are the same at the interface, from the equality of shear stress at the interface we can conclude that the constants of integration $c$ in the three regions are the same, Thus we have

$$\tau_0 R_1^0-\frac{1}{2}\frac{\partial p}{\partial z}(R_1^0)^2=-\tau_0 R_2^0-\frac{1}{2}\frac{\partial p}{\partial z}(R_2^0)^2$$

Simplifying, we get

$$R_2^0-R_1^0=2\tau_0/(-\partial p/z\partial) \tag{2.4}$$

Solving equations (2.2) and (2.3), we have, in the Bingham fluid region adjacent to the inner cylinder,

$$\eta v_z=\frac{-1}{4}\frac{\partial p}{\partial z}(R_1^2-r^2)+\left(R_1^0\tau_0-\frac{1}{2}\frac{\partial p}{\partial z}(R_1^0)^2\right)\ln\frac{r}{R_1}-\tau_0(r-R_1) \tag{2.5}$$

and in the Bingham fluid region adjacent to the outer cylinder,

$$\eta v_z=-\frac{1}{4}\frac{\partial p}{\partial z}(R_2^2-r^2)+\left(-R_2^0\tau_0-\frac{1}{2}\frac{\partial p}{\partial z}(R_2^0)^2\right)\ln\frac{r}{R_2}-\tau_0(R_2-r) \tag{2.6}$$

The positions of the two linking points $R_1^0$ and $R_2^0$ are unknown yet, but there is another relation between them other than (2.4), viz,

$$\begin{aligned}\eta v_z^0&=-4^{-1}\partial p/\partial z(R_2^2-(R_2^0)^2-2(R_2^0)^2\ln[R_2/R_2^0])-\tau_0(R_2-R_2^0-R_2^0\ln[R_2/R_2^0])\\&=-4^{-1}\partial p/\partial z(R_1^2-(R_1^0)^2+2(R_1^0)^2\ln[R_1^0/R_1])-\tau_0(R_1^0-R_1-R_1^0\ln[R_1^0/R_1])\end{aligned} \tag{2.7}$$

From (2.4) and (2.7), the two unknown $R_1^0$ and $R_2^0$ can be solved. Then the quantity of flow $Q$ and the average velocity $\bar{v}_z$ are given by

$$Q\equiv(\pi R_2^2-\pi R_1^2)\bar{v}_z=\int_{R_1}^{R_1^0}2\pi r dr v_z+\pi[(R_2^0)^2-(R_1^0)^2]v_z^0+\int_{R_2^0}^{R_2}2\pi r dr v_z \tag{2.8}$$

Simplifying, we get

$$\eta Q = -\frac{\pi}{8}\frac{\partial p}{\partial z}[(R_2^2-(R_2^0)^2)^2-((R_1^0)^2-R_1^2)^2]$$

$$+\frac{\pi\tau_0}{6}[-2R_2^3-2R_1^3-(R_2^0)^3-(R_1^0)^3+3R_2^0R_2^2+3R_1^0R_1^2] \tag{2.9}$$

**(b) Pipe flow**

The equation of motion in this case is still (2.1), namely

$$-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})=0$$

After integration we have (2.2) again, i.e.

$$\sigma_{rz}=\frac{1}{2}\frac{\partial p}{\partial z}r+\frac{c}{r}$$

The expression for $\sigma_{rz}$ is still

$$\sigma_{rz}=\left(\frac{\tau_0}{|dv_z/dr|}+\eta\right)\frac{dv_z}{dr}$$

The boundary conditions are

$$r=R,\qquad v_z=0$$

At the linking point of Bingham fluid and plugged flow,

$$r=R^0,\ v_z=v_z^0,\ dv_z/dr=0,\ \sigma_{rz}^P=\sigma_{rz}^B$$

Therefore the constants of integration in the two regions are the same. But in the plugged flow region, $\sigma_{rz}$ must be finite, hence $c$ must be zero, That is,

$$-\tau_0=\frac{1}{2}\frac{\partial p}{\partial z}R^0+\frac{c}{R^0}=\frac{1}{2}\frac{\partial p}{\partial z}R^0 \tag{2.10}$$

Thus, we get

$$R_0=2\tau_0/-(\partial p/\partial z) \tag{2.10$'$}$$

In the Bingham fluid region,

$$\eta v_z=\frac{1}{4}\frac{\partial p}{\partial z}(r^2-R^2)+\tau_0(r-R). \tag{2.11}$$

The quantity of flow $Q$ and the average velocity $\bar{v}_z$ are given by

$$\eta Q\equiv\eta\pi R^2\bar{v}=\int_{R_0}^{R}2\pi r\left[\frac{1}{4}\frac{\partial p}{\partial z}(r^2-R^2)+\tau_0(r-R)\right]dr$$

$$+\pi(R^0)^2\left[\frac{1}{4}\frac{\partial p}{\partial z}(R_0^2-R^2)+\tau_0(R^0-R)\right]$$

$$=-\frac{\pi}{8}\frac{\partial p}{\partial z}(R^2-(R^0)^2)^2+\frac{\pi}{6}\tau_0(-2R^3-(R^0)^3+3R^2R^0) \tag{2.12}$$

**(c) Plane pipe flow**

The equation of motion is now

$$-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\sigma_{rz}=0 \tag{2.13}$$

Integrating, we get

$$\sigma_{xy}=\frac{\partial p}{\partial x}y+c \tag{2.14}$$

In the Bingham fluid region, the expression for $\sigma_{xy}$ is

$$\sigma_{xy}=\left(\frac{\tau_0}{|dv_x/dy|}+\eta\right)\frac{dv_x}{dy} \tag{2.15}$$

The boundary conditions are
At the walls, $y=\pm h,\quad v_x=0$
At the interface with plugged flow,

$$y=\pm b,\ v_x=v_x^0,\ dv_x/dy=0,\ \sigma_{xy}^P=\sigma_{xy}^B,$$

Therefore in the two Bingham fluid regions and the plugged flow region, the constants of integration $c$ are the same. Thus,

$$-\tau_0=\frac{\partial p}{\partial x}b+c,\quad \tau_0=-\frac{\partial p}{\partial x}b+c$$

Thereby

$$c=0,\qquad b=\frac{\tau_0}{-\partial q/\partial x} \tag{2.16}$$

The velocity distributions in the Bingham fluid regions are

$$y\geqslant b,\ \eta v_x=-\frac{1}{2}\frac{\partial p}{\partial x}(h^2-y^2)+\frac{\partial p}{\partial x}b(h-y)$$

$$y\leqslant -b,\ \eta v_x=-\frac{1}{2}\frac{\partial p}{\partial x}(h^2-y^2)+\frac{\partial p}{\partial x}b(h+y)$$

The quantity of flow $Q$ and the average velocity are given by

$$\eta Q\equiv 2\eta h\bar{v}_x=2\int_b^h -\frac{1}{2}\frac{\partial p}{\partial x}[h^2-y^2-2b(h-y)]dy$$

$$+2b\left(-\frac{1}{2}\frac{\partial p}{\partial x}\right)[h^2-b^2-2b(h-b)]$$

$$=-\frac{\partial p}{\partial x}\left[\frac{2}{3}h^3+\frac{1}{3}b^3-bh^2\right] \tag{2.17}$$

## III. Approximate Solutions in Several Special Cases

Let

$$\lambda\equiv\frac{\tau_0}{-\partial p/\partial z}\left(\text{or}\ \frac{\tau_0}{-\partial p/\partial x}\right) \tag{3.1}$$

**(a) Annular flow**

The equations determining $R_1^0$ and $R_2^0$ now become

$$\begin{cases} R_2^2-(R_2^0)^2-2(R_2^0)^2\ln\dfrac{R_2}{R_2^0}-4\lambda\left(R_2-R_2^0-R_2^0\ln\dfrac{R_2}{R_2^0}\right) \\ \quad =R_1^2-(R_1^0)^2+2(R_1^0)^2\ln\dfrac{R_1^0}{R_1}-4\lambda\left(R_1^0-R_1-R_1^0\ln\dfrac{R_1^0}{R_1}\right) & (3.2) \\ R_2^0-R_1^0=2\lambda & (3.3) \end{cases}$$

Before solving (3.2) and (3.3), Let us consider the case of Newtonian fluid. In this case $R_1$ coincides with $R_2$ and equals the radius for maximum velocity, $R_m$. The equation satisfied by $R_m$ is

$$R_2^2-R_m^2-2R_m^2\ln\frac{R_2}{R_m}=R_1^2-R_m^2+2R_m^2\ln\frac{R_m}{R_1} \tag{3.4}$$

Solving $R_m$, we get

$$R_m=\sqrt{\frac{R_2^2-R_1^2}{2\ln(R_2/R_1)}} \tag{3.5}$$

Let

$$R_2^0=R_m+k_2\lambda,\quad R_1^0=R_m-k_1\lambda \tag{3.6}$$

Thus

$$k_1+k_2=2 \tag{3.7}$$

When $\lambda$ is very small, we may neglect higher order terms in $\lambda$. Solving (3.2) and (3.3)

$$\left.\begin{aligned} k_1&=\left(\frac{R_1+R_2}{R_m}-2+\ln\frac{R_2}{R_1}\right)\Big/\ln\frac{R_2}{R_1} \\ k_2&=\left(\ln\frac{R_2}{R_1}+2-\frac{R_1+R_2}{R_m}\right)\Big/\ln\frac{R_2}{R_1} \end{aligned}\right\} \tag{3.8}$$

In order to compare with the commonly used formula in petroleum engineering, we have for the case when $R_2-R_1\ll R_1$,

$$\begin{aligned} \eta Q&\equiv\eta\pi(R_2^2-R_1^2)\bar{v}_z\approx-\frac{\pi}{8}\frac{\partial p}{\partial z}[(R_2^2-R_m^2)^2-(R_m^2-R_1^2)^2] \\ &+\frac{\pi}{6}\tau_0[-2R_2^3-2R_1^3-R_m^2(2-3k_2-3k_1) \\ &+3R_mR_2^2(1-k_2)+3R_mR_1^2(1-k_1)] \end{aligned} \tag{3.9}$$

And we have

$$\left.\begin{aligned} R_m&=\sqrt{\frac{R_2^2-R_1^2}{2\ln(R_2/R_1)}}\approx\sqrt{\frac{R_1(R_1+R_2)}{2}}\left(1+\frac{1}{4}\,\frac{R_2-R_1}{R_1}\right) \\ &\approx R_1\left(1+\frac{1}{2}\,\frac{R_2-R_1}{R_1}\right)\approx\frac{1}{2}(R_1+R_2) \\ 2R_m^2&\approx\frac{1}{2}(R_1+R_2)^2,\quad k_1\approx k_2\approx 1 \\ R_2^2&+R_1^2-2R_m^2\approx\frac{1}{2}(R_2-R_1)^2 \end{aligned}\right\} \tag{3.10}$$

Simplifying (3.9) by means of (3.10), we get

$$-\frac{\partial p}{\partial z}=\frac{16\eta\bar{v}_z}{(R_2-R_1)^2}+\frac{4\tau_0}{R_2-R_1} \tag{3.11}$$

Replacing $R$ by diameter $D$,

$$-\frac{\partial p}{\partial z}=\frac{64\eta\bar{v}_z}{(D_2-D_1)^2}+\frac{8\tau_0}{D_2-D_1} \tag{3.11$'$}$$

**(b) Pipe flow**

Neglecting higher order terms in $\lambda$, we have now

$$\eta Q\equiv\eta\pi R^2\bar{v}_z\approx-\frac{\pi}{8}\frac{\partial p}{\partial z}R^4-\frac{\pi}{3}\tau_0 R^3 \tag{3.12}$$

After simplification we get

$$-\frac{\partial p}{\partial z}=\frac{8\eta\bar{v}_z}{R^2}+\frac{8}{3}\frac{\tau_0}{R} \tag{3.13}$$

Replacing $R$ by diameter $D$,

$$-\frac{\partial p}{\partial z}=\frac{32\eta\bar{v}_z}{D^2}+\frac{16}{3}\frac{\tau_0}{D} \tag{3.13$'$}$$

**(c) Plane pipe flow**

Neglecting higher order terms in $\lambda$, we have

$$\eta Q\equiv 2h\eta\bar{v}_x=-\frac{\partial p}{\partial x}\left[\frac{2}{3}h^3-\lambda h^2\right]=-\frac{2}{3}h^3\frac{\partial p}{\partial x}-\tau_0 h^2 \tag{3.14}$$

Simplifying, we get

$$-\frac{\partial p}{\partial x}=\frac{3\eta\bar{v}_x}{h^2}+\frac{3\tau_0}{2h} \tag{3.15}$$

## IV. Comparison with the Commonly Used Formula

In the petroleum industry, the rigidity modulus and the yielding stress are often denoted by the readings of viscosimeter, $\theta$. $\theta_{600}$ means the viscosimeter reading at 600 rpm, $\theta_{300}$ means the viscosimeter reading at 300 rpm. The rigidity modulus $(pv)=\theta_{600}-\theta_{300}$, the yielding stress $y=\theta_{300}-(pv)$. For annular flow, the commonly used formula for pressure drop[1] is

$$p=\frac{(pv)\bar{v}l}{60000(D_h-D_p)^2}+\frac{yl}{200(D_h-D_p)} \tag{4.1}$$

where $D_h$ is the diameter of the outer tube, $D_p$ the diameter of the inner tube, $\bar{v}$ is the average velocity, $l$ is the well depth, $P$ is the pressure drop. Formula (4.1) is identical in form with (3.11)$'$.

For the case of pipe flow, the common formula for pressure drop is

$$p=\frac{(pv)\bar{v}l}{90000D^2}+\frac{yl}{225D} \tag{4.2}$$

Formula (4.2) is also identical in form with (3.13)$'$.

### Reference

[1] Moore, P.L., *Drilling Practices Manual.* The Petroleum Publishing Co., U.S.A. (1974).