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THE BOUNDARY LAYER SCHEME FOR A SINGULARLY PERTURBED PROBLEM FOR THE SECOND ORDER ELLIPTIC EQUATION IN THE RECTANGLE

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Abstract

Using singularly perturbation theory is constructed the boundary layer scheme for a Dirichlet problem for the second order singularly perturbed equation of elliptic type in the rectangle. The error estimate is given.

I. Introduction

In the rectangle $G+\Gamma$: $(0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b)$ we consider the following Dirichlet problem for the second order elliptic equation:

$$\mathscr{L}_\varepsilon u\equiv\varepsilon^2\Delta u-\frac{\partial u}{\partial y}-u=f(x,y)\qquad\left((x,y)\in G=\left(\begin{matrix}0<x<a\\0<y<b\end{matrix}\right)\right)\tag{1.1}$$

$$u|_\Gamma=0\tag{1.2}$$

where $\varepsilon>0$ is a small parameter. As $\varepsilon=0$ the corresponding reduced problem is

$$\mathscr{L}_0 w\equiv-\frac{\partial w}{\partial y}-w=f(x,y)\tag{1.3}$$

$$w|_{y=0}=0\tag{1.4}$$

Therefore, when perturbed problems (1.1), (1.2) are degenerated to problems (1.3), (1.4) boundary conditions will be lost at the boundaries $x=0$, $x=a$ and $y=b$. Nearby them the boundary layer will arise. It is known that $x=$ const. are the characteristics of reduced equation (1.3). So problems (1.1), (1.2) belong to the problems with characteristic boundaries. Following the singular perturbation theory nearby the boundaries $x=0$ and $x=a$ will occur parabolic boundary layers. Emelyanov[2],[3], Miller[4] investigated the difference methods for solving this problem. But they excluded subdomains, in which occur parabolic boundary layers, and considered only the case of non-characteristic boundary. It seems that construction of exponentially fitting difference schemes is very difficult. Following Hsio and Jordan[5] in this paper we consider the general boundary layer, parabolic boundary layers and corner boundary layers at points (0, *b*) and (*a*,*b*), and construct the boundary layer scheme for problems (1.1), (1.2).

Assume function $f(x,y)$ is sufficiently smooth in $G+\Gamma$ and satisfies

$$f(0,0)=0,\ \ f(a,0)=0\tag{1.5}$$

II. Construction of the Asymptotic Solutions

We construct the asymptotic expansion of the solution $u(x,y,\varepsilon)$ of problems (1.1), (1.2) in the following:

$$\tilde{u}(x,y,\varepsilon)=w(x,y)+v^{(0)}(\xi_1,y)+v^{(1)}(\xi_2,y)+v^{(2)}(x,\eta)$$
$$+v^{(3)}(\xi_1,\eta)+v^{(4)}(\xi_2,\eta)+O(\varepsilon) \qquad (2.1)$$

where 1) $\xi_1=x/\varepsilon,\ \ \xi_2=(a-x)/\varepsilon,\ \ \eta=(b-y)/\varepsilon^2$ (2.2)

2) $w(x,y)$ is the solution of reduced problems (1.3), (1.4)

3) $v^{(0)}\ (\xi_1,y)$ —boundary layer function which is constructed nearby boundary ($x=0,\ \ 0\leqslant y\leqslant b$), and satisfies

$$M_0v^{(0)}\equiv\frac{\partial^2 v^{(0)}}{\partial\xi_1^2}-\frac{\partial v^{(0)}}{\partial y}-v^{(0)}=0 \qquad \left(\begin{matrix}0<\xi_1<\infty\\ 0<y<b\end{matrix}\right) \qquad (2.3)$$

$$\left.\begin{matrix}v^{(0)}(\xi_1,0)=0,\ \ v^{(0)}(0,y)=-w(0,y)\\ v^{(0)}(\xi_1,y)\to 0\quad(\xi_1\to\infty)\end{matrix}\right\} \qquad (2.4)$$

This is the first boundary value problem for parabolic equation in the semi-infinite region. The exact solution of this problem is

$$v^{(0)}(\xi_1,y)=\frac{\exp(-y)}{2\sqrt{\pi}}\int_0^y\frac{\xi_1}{(y-\tau)^{3/2}}\exp\left[-\frac{\xi_1^2}{4(y-\tau)}\right]\varphi(\tau)d\tau$$
$$\varphi(\tau)=-\exp(-\tau)w(0,\tau) \qquad (2.5)$$

Expression (2.5) can be rewritten as

$$v^{(0)}(\xi_1,y)=\frac{2\exp(-y)}{\sqrt{\pi}}\int_{\frac{\xi_1}{2\sqrt{y}}}^{\infty}\exp(-s^2)\varphi\left(y-\frac{\xi_1^2}{4s^2}\right)ds \qquad (2.6)$$

From condition (1.5) straightforward analysis shows that $\partial v^{(0)}/\partial y,\ \ \partial^2v^{(0)}/\partial y^2$ are bounded in the neighborhood of corner point (0,0).

4) $v^{(1)}(\xi_2,y)$ —boundary layer function constructed nearby boundary ($x=a,\ \ 0\leqslant y\leqslant b$) :

$$v^{(1)}(\xi_2,y)=\frac{2\exp(-y)}{\sqrt{\pi}}\int_{\frac{\xi_2}{2\sqrt{\pi}}}^{\infty}\exp(-s^2)\varphi\left(y-\frac{\xi_2^2}{4s^2}\right)ds \qquad (2.7)$$

$$\varphi(\tau)=-\exp(-\tau)w(a,\tau)$$

From condition (1.5) the derivatives $\partial v^{(1)}/\partial y,\ \ \partial^2v^{(1)}/\partial y^2$ are bounded in the neighborhood of corner point (a,0).

Since for arbitrary constant $k>0$ hold $\exp(-s^2)\ \leqslant\exp\ (k^2/4)\exp(-ks)=c\exp(-ks)$ and $|\varphi(\tau)|\leqslant c$, from (2.6), (2.7)
We obtain

$$|v^{(0)}(\xi_1,y)|\leqslant c\exp(-\alpha_1\xi_1)\quad(\alpha_1>0) \qquad (2.8)$$
$$|v^{(1)}(\xi_2,y)|\leqslant c\exp(-\alpha_2\xi_2)\quad(\alpha_2>0) \qquad (2.9)$$

5) $v^{(2)}(x,\eta)$ —boundary layer function constructed nearby boundary

$(y=b,\ \ 0\leqslant x\leqslant a)$, which satisfies

$$Nv^{(2)}(x,\eta)\equiv\frac{\partial^2 v^{(2)}}{\partial\eta^2}+\frac{\partial v^{(2)}}{\partial\eta}=0 \tag{2.10}$$

$$v^{(2)}(x,0)=-w(x,b),\ \ v^{(2)}(x,\eta)\to 0\quad(\eta\to\infty), \tag{2.11}$$

Its analytic expression is

$$v^{(2)}(x,\eta)=-w(x,b)\exp(-\eta) \tag{2.12}$$

and it has the estimate

$$|v^{(2)}(x,\eta)|\leqslant c\exp(-\eta) \tag{2.13}$$

6) $v^{(3)}(\xi_1,\eta)$ —boundary layer function defind in the neighborhood of a corner point (0,*b*), which satisfies

$$R_0 v^{(3)}\equiv\frac{\partial^2 v^{(3)}}{\partial\eta^2}+\frac{\partial v^{(3)}}{\partial\eta}=0\qquad\begin{pmatrix}0<\xi_1<\infty\\ 0<\eta<\infty\end{pmatrix} \tag{2.14}$$

$$v^{(3)}(\xi_1,0)=-v^{(0)}(\xi_1,b),\ \ v^{(3)}(\xi_1,\eta)\to 0\quad(\eta\to\infty) \tag{2.15}$$

It can be explicitly represented as

$$v^{(3)}(\xi_1,\eta)=-v^{(0)}(\xi_1,b)\exp(-\eta) \tag{2.16}$$

and has the estimete

$$|v^{(3)}(\xi_1,\eta)|\leqslant c\exp(-a_3(\xi_1+\eta)) \tag{2.17}$$

We construct $v^{(3)}(\xi_1,\eta)$ for the reason that $v^{(0)}(\xi_1,y)$ does not satisfy the boundary condition at $y=b$.

7) $v^{(4)}(\xi_2,\eta)$—boundary layer function defined in the neighborhood of corner point (*a*,*b*):

$$v^{(4)}(\xi_2,\eta)=-v^{(1)}(\xi_2,b)\exp(-\eta) \tag{2.18}$$

$$|v^{(4)}(\xi_2,\eta)|\leqslant c\exp(-a_4(\xi_2+\eta)) \tag{2.19}$$

Analogously, we construct $v^{(4)}(\xi_2,\eta)$ for the reason that $v^{(1)}(\xi_2,y)$ does not satisfy the boundary condition at $y=b$.

If we expect that the accuracy of the asymptotic solution can be extended to the order $O(\varepsilon^{n+1})$ $(n\geqslant 2)$, it is necessary to consider the cases that the sums $v^{(2)}(x,\eta)+v^{(3)}(\xi_1,\eta)$ and $v^{(2)}(x,\eta)+v^{(4)}(\xi_2,\eta)$ do not satisfy the boundary conditions at $x=0$ and $x=a$ respectively. However, as $n=0,1$ these boundary conditions are satisfied (cf. Butuzov[6]) Butuzov proves that expansion (2.1) uniformly holds in $G+\Gamma$. Let

$$\begin{aligned}\tilde{u}(x,y,\varepsilon)=&w(x,y)+v^{(0)}(\xi_1,y)+v^{(1)}(\xi_2,y)+v^{(2)}(x,\eta)\\&+v^{(3)}(\xi_1,\eta)+v^{(4)}(\xi_2,\eta)\end{aligned} \tag{2.20}$$

then

$$u(x,y,\varepsilon)-\tilde{u}(x,y,\varepsilon)=O(\varepsilon)\qquad((x,y)\in G+\Gamma) \tag{2.21}$$

III. The Construction of Difference Schemes and the Error Estimates

1. The difference scheme of the reduced problem

We use the modified Euler's method to solve reduced problems (1.3), (1.4). Let h and k denote the mesh spacings in the x-direction and y-direction respectively, $x=x_i=ih$, $y=y_j=jk$, $i=0,1,\ldots,N$, $j=0,1,\ldots,J$, $Nh=a$, $Jk=b$.

Now we construct the following scheme:

$$\frac{1}{k}\left[w^{(h,k)}(x,y+k)-w^{(h,k)}(x,y)\right]+\frac{1}{2}[w^{(h,k)}(x,y+k)+w^{(h,k)}(x,y)]$$

$$=\frac{-1}{2}[f(x,y)+f(x,y+k)] \tag{3.1}$$

$$w^{(h,k)}(x,0)=0 \tag{3.2}$$

This is a scheme of second order:

$$|w(x,y)-w^{(h,k)}(x,y)|=O(k^2) \qquad (0\leqslant x_i\leqslant a;\ 0\leqslant y_j\leqslant b) \tag{3.3}$$

2. The difference schemes of the boundary layer equations

In our problem the boundary layer equations are always solved in semi-infinite regions. If we directly solve them by difference methods, then this requires considerable computational effort, because of the presence of ε. But from the singular perturbation theory we know that the boundary layer functions are significant only in the boundary layer regions. Following the work[5] we numerically solve these equations by the standard difference methods only in the finite regions. For this it is necessary to modify the problems for the boundary layer equations.

1) The modification of problems (2.3), (2.4).

Find $\tilde{v}^{(0)}(\xi_1,y)$:

$$M_0\tilde{v}^{(0)}\equiv\frac{\partial^2\tilde{v}^{(0)}}{\partial\xi_1^2}-\frac{\partial\tilde{v}^{(0)}}{\partial y}-\tilde{v}^{(0)}=0 \qquad \begin{pmatrix}0<\xi_1<m_1\\ 0<y\leqslant b\end{pmatrix} \tag{3.4}$$

$$\left.\begin{aligned}&\tilde{v}^{(0)}(\xi_1,0)=0 && (0\leqslant\xi_1\leqslant m_1)\\ &\tilde{v}^{(0)}(0,y)=-w(0,y),\quad \tilde{v}^{(0)}(m_1,y)=0 && (0\leqslant y\leqslant b)\end{aligned}\right\}. \tag{3.5}$$

where $m_1>0$ is a constant to be determined.

It is known that the solution (2.5) of problems (2.3), (2.4) are a boundary layer function, and holds estimete (2.8). We choose m_1 so that $\exp(-\alpha_1 m_1)\leqslant\varepsilon$, and obtain

$$m_1\geqslant-\frac{1}{\alpha_1}\ln\varepsilon \tag{3.6}$$

It is easily shown that this choice of m_1 will not cause an error higher order $O(\varepsilon)$, i.e.

$$|v^{(0)}(\xi_1,y)-\tilde{v}^{(0)}(\xi_1,y)|=O(\varepsilon) \tag{3.7}$$

Problems (3.4), (3.5) are the first boundary value problem for parabolic equation in the finite region. For it we can construct Crank-Nicolson scheme:

$$M_0^{(\tilde{h}_1,k)}\tilde{v}^{(0)(\tilde{h}_1,h)}\equiv\frac{1}{2}\left[\tilde{v}^{(0)(\tilde{h}_1,k)}_{\xi_1\xi_1}(\xi_1,y+k)+\tilde{v}^{(0)(\tilde{h}_1,k)}_{\xi_1\xi_1}(\xi_1,y)\right]$$

$$-\tilde{v}_y^{(0)(\tilde{h}_1,k)}(\xi_1,y)-[\tilde{v}^{(0)(\tilde{h}_1,k)}(\xi_1,y+k)+\tilde{v}^{(0)(\tilde{h}_1,k)}(\xi_1,y)]/2=0 \tag{3.8}$$

$$\tilde{v}^{(0)(\tilde{h}_1,k)}(\xi_1,0)=0,\quad \tilde{v}^{(0)(\tilde{h}_1,k)}(0,y)=-w^{(h,k)}(0,y)$$
$$\tilde{v}^{(0)(\tilde{h}_1,k)}(m_1,y)=0 \tag{3.9}$$

where $\xi_1=\xi_{1,i}=i\tilde{h}_1,\ y=y_j=jk,\ \tilde{h}_1=m_1/N_1,\ k=b/J,\ \tilde{v}^{(0)(\tilde{h}_1,k)}_{\xi_1\bar{\xi}_1}$ —the centered second difference, $\tilde{v}_y^{(0)(\tilde{h}_1,k)}=\frac{1}{k}[\tilde{v}^{(0)(\tilde{h}_1,k)}(x,y+k)-\tilde{v}^{(0)(\tilde{h}_1,k)}(x,y)]$.

It is known that schemes (3.8), (3.9) are of second order, i.e.

$$|\tilde{v}^{(0)(\tilde{h}_1,k)}(\xi_1,y)-\tilde{v}^{(0)}(\xi_1,y)|=O(\tilde{h}_1^2+k^2) \tag{3.10}$$

However, one does not know a priori the value of-$w(0,y)$ in boundary condition (3.9), and hence one is acturally using the value $-w^{(h,k)}(0,y)$. This, of course introduces additional error. From (3.3) $|w(0,y)-w^{(h,k)}(0,y)|=O(k^2)$. Note that here $\xi_1=\xi_{1,i}$ corresponds to x_i, i.e. $\xi_{1,i}=x_i/\varepsilon$

2) The modification of differential problem defining $v^{(1)}(\xi_2,y)$

Find $\tilde{v}^{(1)}(\xi_2,y)$:

$$M_1\tilde{v}^{(1)}(\xi_2,y)\equiv\frac{\partial^2\tilde{v}^{(1)}}{\partial\xi_2^2}-\frac{\partial\tilde{v}^{(1)}}{\partial y}-\tilde{v}^{(1)}=0 \quad \left(\begin{array}{c}0<\xi_2<m_2\\0<y<b\end{array}\right) \tag{3.11}$$

$$\left.\begin{array}{ll}\tilde{v}^{(1)}(\xi_2,0)=0 & (0\leqslant\xi_2\leqslant m_2)\\ \tilde{v}^{(1)}(0,y)=-w(a,y),\ \tilde{v}^{(1)}(m_2,y)=0 & (0\leqslant y\leqslant b)\end{array}\right\} \tag{3.12}$$

where $m_2>0$ is a constant to be determined. We know that $\tilde{v}^{(1)}(\xi_2,y)$ has the similar estimate (2.9). Analogously, m_2 is defined as

$$m_2\geqslant-\frac{1}{\alpha_2}\ln\varepsilon \tag{3.13}$$

and

$$|v^{(1)}(\xi_2,y)-\tilde{v}^{(1)}(\xi_2,y)|=O(\varepsilon) \tag{3.14}$$

For problems (3.11), (3.12) we also construct Crank-Nicolson scheme:

$$M_1^{(\tilde{h}_2,k)}\tilde{v}^{(1)(\tilde{h}_2,k)}\equiv\frac{1}{2}[\tilde{v}^{(1)(\tilde{h}_2,k)}_{\xi_2\bar{\xi}_2}(\xi_2,y+k)+\tilde{v}^{(1)(\tilde{h}_2,k)}_{\xi_2\bar{\xi}_2}(\xi_2,y)]$$
$$-\tilde{v}_y^{(1)(\tilde{h}_2,k)}(\xi_2,y)-\frac{1}{2}[\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,y+k)+\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,y)]=0 \tag{3.15}$$

$$\left.\begin{array}{l}\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,0)=0\\ \tilde{v}^{(1)(\tilde{h}_2,k)}(0,y)\doteq-w^{(h,k)}(a,y),\ \tilde{v}^{(1)(\tilde{h}_2,k)}(m_2,y)=0\end{array}\right\} \tag{3.16}$$

where $\xi_2=\xi_{2,i}=i\tilde{h}_2,\ y_j=jk,\ N_2\tilde{h}_2=m_2,\ Jk=b$. In the boundary condition (3.16) instead of $-w(a,y)$ we take the value $-w^{(h,k)}(a,y)$. From (3.3) $|w(a,y)-w^{(h,k)}(a,y)|=O(k^2)$. Hence we obtain

$$|\tilde{v}^{(1)}(\xi_2,y)-\tilde{v}^{(1)(\tilde{h},k)}(\xi_2,y)|=O(\tilde{h}_2^2+k^2) \tag{3.17}$$

where $\xi_2=\xi_{2,i}=(a-x_i)/\varepsilon$.

3) The modification of problems (2.10), (2.11).

Find $\tilde{v}^{(2)}(x,\eta)$:

$$N\tilde{v}^{(2)}(x,\eta)\equiv\frac{\partial^2\tilde{v}^{(2)}}{\partial\eta^2}+\frac{\partial\tilde{v}^{(2)}}{\partial\eta}=0 \qquad \begin{pmatrix}0<x<a\\0<\eta<s\end{pmatrix} \tag{3.18}$$

$$\tilde{v}^{(2)}(x,0)=-w(x,b);\ \tilde{v}^{(2)}(x,s)=0 \qquad (0\leqslant x\leqslant a) \tag{3.19}$$

where $s>0$ is a constant to be determined.

It is known that holds the estimate (2.13) for the solution of problems (2.10), (2.11). From this the constant s can be defined as

$$s\geqslant-2\ln\varepsilon \tag{3,20}$$

and

$$|\tilde{v}^{(2)}(x,\eta)-v^{(2)}(x,\eta)|=O(\varepsilon^2) \tag{3.21}$$

Problems (3.18), (3.19) are the two-point boundary value problem for ordinary differential equation. Now is given the following difference scheme:

$$N^{(h,\tilde{k})}\tilde{v}^{(2)(h,\tilde{k})}\equiv\tilde{v}^{(2)(h,\tilde{k})}_{\eta\bar{\eta}}-\tilde{v}^{(2)(h,\tilde{k})}_{\hat{\eta}}=0 \tag{3.22}$$

$$\tilde{v}^{(2)(h,\tilde{k})}(x,0)=-w^{(h,k)}(x,b),\ \ \tilde{v}^{(2)(h,\tilde{k})}(x,s)=0 \tag{3.23}$$

where $\eta=\eta_j=j\tilde{k}$, $\tilde{J}\tilde{k}=s$, $x=x_i=ih$, $Nh=a$. Instead of $-w(x,b)$ we take $-w^{(h,k)}(x,y)$ From (3.3) $|w(x,b)-w^{(h,k)}(x,b)|=O(k^2)$. Thus

$$|\tilde{v}^{(h,\tilde{k})}(x,\eta)-\tilde{v}(x,\eta)|=O(k^2+\tilde{k}^2) \tag{3.24}$$

Here $\eta=\eta_j=(b-y_j)/\varepsilon^2$

4) The modification of problems (2.14), (2.15).

Find $\tilde{v}^{(3)}(\xi_1,\eta)$:

$$R_0\tilde{v}^{(3)}\equiv\frac{\partial^2\tilde{v}^{(3)}}{\partial\eta^2}+\frac{\partial\tilde{v}^{(3)}}{\partial\eta} \qquad \begin{pmatrix}0<\eta<s\\0<\xi_1<m_1\end{pmatrix} \tag{3.25}$$

$$\tilde{v}^{(3)}(\xi_1,0)=-v^{(0)}(\xi_1,b),\ \ \tilde{v}^{(3)}(\xi_1,s)=0 \quad (0\leqslant\xi_1\leqslant m_1) \tag{3.26}$$

The solution (2.16) of problems (2.14), (2.15) has the estimate (2.17). Constants m_1 and s defined by (3.6) and (3.20) are respectively appropriate for this problem, only if constant α_3 is properly chosen, and

$$|v^{(3)}(\xi_1,\eta)-\tilde{v}^{(3)}(\xi_1,\eta)|=O(\varepsilon^2) \tag{3.27}$$

For (3.25), (3.26) we give the following difference scheme:

$$R_0^{(\tilde{k}_1,\tilde{k})}\tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}\equiv\tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}_{\eta\bar{\eta}}+\tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}_{\hat{\eta}}=0 \tag{3.28}$$

$$\tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}(\xi_1,0)=-\tilde{v}^{(0)(\tilde{k}_1,\tilde{k})}(\xi_1,b),\ \ \tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}(\xi_1,s)=0 \tag{3.29}$$

where $\eta=\eta_j=j\tilde{k}$, $\xi_1=\xi_{1,i}=i\tilde{k}_1$, $\eta_j=(b-y_j)/\varepsilon^2$, $\xi_{1,i}=x_i/\varepsilon$. In the boundary condition (3.29) we take $-\tilde{v}^{(0)(\tilde{k}_1,k)}(\xi_1,b)$ instead of $-v^{(0)}(\xi_1,b)$. From this $|v^{(0)}(\xi_1,b)-\tilde{v}^{(0)(\tilde{k}_1,k)}(\xi_1,b)|=O(\varepsilon)+O(\tilde{k}_1^2+k^2)$. Hence

$$|\tilde{v}^{(3)(\tilde{k}_1,\tilde{k})}(\xi_1,\eta)-\tilde{v}^{(3)}(\xi_1,\eta)|=O(\tilde{k}^2)+O(\tilde{k}_1^2+k^2)+O(\varepsilon) \tag{3.30}$$

5) The modification of the differential problem defining $v^{(4)}(\xi_2,\eta)$.

Find $\tilde{v}^{(4)}(\xi_2,\eta)$:

$$R_1\tilde{v}^{(4)} \equiv \frac{\partial^2 \tilde{v}^{(4)}}{\partial \eta^2} + \frac{\partial \tilde{v}^{(4)}}{\partial \eta} = 0 \qquad \begin{pmatrix} 0<\xi_2<m_2 \\ 0<\eta<s \end{pmatrix} \tag{3.31}$$

$$\tilde{v}^{(4)}(\xi_2,0) = -v^{(1)}(\xi_2,b), \quad \tilde{v}^{(4)}(\xi_2,s)=0 \qquad (0\leqslant\xi_2\leqslant m_2) \tag{3.32}$$

From (2.19) constants m_2 and s defined by (3.13) and (3.26) are also respectively appropriate for this problem in the case of proper choice of constant α_4 , and

$$|\tilde{v}^{(4)}(\xi_2,\eta) - v^{(4)}(\xi_2,\eta)| = O(\varepsilon^2) \tag{3.33}$$

For (3.31), (3.32) the difference scheme is given by

$$R_1(\tilde{h}_2,\tilde{k})\tilde{v}^{(4)(\tilde{h}_2,\tilde{k})} \equiv \tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}_{\eta\bar{\eta}} + \tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}_{\mathring{\eta}} = 0 \tag{3.34}$$

$$\tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}(\xi_2,0) = -\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,b), \quad \tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}(\xi_2,s)=0 \tag{3.35}$$

where $\eta=\eta_j=j\tilde{k}$, $\xi_2=\xi_{2,i}=i\tilde{h}_2$, $\eta_j=(b-y_j)/\varepsilon^2$, $\xi_{2,i}=(a-x_i)$
Similarly, in the boundary condition (3.35) the value $-\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,b)$ is taken instead of $-v^{(1)}(\xi_2,b)$. This causes error $|v^{(1)}(\xi_2,b)-\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,b)|=O(\tilde{h}_2^2+k^2)+O(\varepsilon)$. Hence

$$|\tilde{v}^{(4)}(\xi_2,\eta) - \tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}(\xi_2,\eta)| = O(\tilde{k}^2) + O(\tilde{h}_2^2+\bar{k}^2) + O(\varepsilon) \tag{3.36}$$

Solving difference problems (3.1), (3.2); (3.8), (3.9); (3.11). (3.12), (3.22). (3.23); (3.28), (3.29) and (3.34), (3.35) respectively, we obtain the numerical results for perturbed problems (1.1), (1.2):

$$u^{(h,k)}(x,y,\varepsilon) = \begin{cases} w^{(h,k)}(x,y) & \begin{pmatrix} \varepsilon m_1\leqslant x\leqslant a-\varepsilon m_2 \\ 0\leqslant y\leqslant b-\varepsilon^2 s \end{pmatrix} \\ w^{(h,k)}(x,y)+\tilde{v}^{(0)(\tilde{h}_1,k)}(\xi_1,y) & \begin{pmatrix} 0\leqslant\xi_1\leqslant m_1 \\ 0\leqslant y\leqslant b-\varepsilon^2 s \end{pmatrix}, \quad x=\varepsilon\xi_{1,i} \\ w^{(h,k)}(x,y)+\tilde{v}^{(1)(\tilde{h}_2,k)}(\xi_2,y) & \begin{pmatrix} 0\leqslant\xi_2\leqslant m_2) \\ 0\leqslant y\leqslant b-\varepsilon^2 s \end{pmatrix}, \quad x=\varepsilon\xi_{2,i} \\ w^{(h,\bar{k})}(x,y)+\tilde{v}^{(2)(h,\bar{k})}(x,\eta) & \begin{pmatrix} \varepsilon m_1\leqslant x\leqslant a-\varepsilon m_2 \\ 0\leqslant\eta\leqslant s \end{pmatrix}, y=b-\varepsilon^2\eta_j \\ w^{(\bar{h}_1,\bar{k})}(x,y)+\tilde{v}^{(3)(\tilde{h}_1,\tilde{k})}(\xi_1,\eta) & \begin{pmatrix} 0\leqslant\xi_1\leqslant m_1 \\ 0\leqslant\eta\leqslant s \end{pmatrix}, \quad \begin{matrix} x=\varepsilon\xi_{1,i} \\ y=b-\varepsilon^2\eta_j \end{matrix} \\ w^{(\bar{h}_2,\bar{k})}(x,y)+\tilde{v}^{(4)(\tilde{h}_2,\tilde{k})}(\xi_2,\eta) & \begin{pmatrix} 0\leqslant\xi_2\leqslant m_2 \\ 0\leqslant\eta\leqslant s \end{pmatrix}, \quad \begin{matrix} x=a-\varepsilon\xi_{2,i} \\ y=b-\varepsilon^2\eta_j \end{matrix} \end{cases} \tag{3.37}$$

here

$$\bar{k}=\varepsilon^2\tilde{k}, \quad \bar{h}_1=\varepsilon\tilde{h}_1, \quad \bar{h}_2=\varepsilon\tilde{h}_2 \tag{3.38}$$

h, k, $\tilde{k}$, $\tilde{h}_1$, $\tilde{h}_2$ are defined by $h=a/N$, $k=b/J$, $\tilde{k}=s/\tilde{J}$, $\tilde{h}_1=m_1/N_1$, $\tilde{h}_2=m_2/N_2$ respectively.

From the asymptotic expansion (2.1), estimates (3.3), (3.7), (3.10), (3.14), (3.17), (3.21), (3.24), (3.27), (3.30), (3.33), (3.36), (3.37) and (3.38) we easily obtain the following error estimate:

$$u(x,y,\varepsilon)-u^{(h,k)}(x,y,\varepsilon)=\left\{\begin{array}{ll} O(k^2)+O(\varepsilon) & \left(\begin{array}{l}\varepsilon m_1\leqslant x\leqslant a-\varepsilon m_2\\ 0\leqslant y\leqslant b-\varepsilon^2 s\end{array}\right)\\ O(\tilde{h}_1^2+k^2)+O(\varepsilon) & \left(\begin{array}{l}0\leqslant x\leqslant \varepsilon m_1\\ 0\leqslant y\leqslant b-\varepsilon^2 s\end{array}\right)\\ O(\tilde{h}_2^2+k^2)+O(\varepsilon) & \left(\begin{array}{l}a-\varepsilon m_2\leqslant x\leqslant a\\ 0\leqslant y\leqslant b-\varepsilon^2 s\end{array}\right)\\ O(\tilde{k}^2)+O(\varepsilon) & \left(\begin{array}{l}\varepsilon m_1\leqslant x\leqslant a-\varepsilon m_2\\ b-\varepsilon^2 s\leqslant y\leqslant b\end{array}\right)\\ O(\tilde{h}_1^2+\tilde{k}^2)+O(k^2)+O(\varepsilon) & \left(\begin{array}{l}0\leqslant x\leqslant \varepsilon m_1\\ b-\varepsilon^2 s\leqslant y\leqslant b\end{array}\right)\\ O(\tilde{h}_2^2+\tilde{k}^2)+O(k^2)+O(\varepsilon) & \left(\begin{array}{l}a-\varepsilon m_2\leqslant x\leqslant a\\ b-\varepsilon^2 s\leqslant y\leqslant b\end{array}\right)\end{array}\right\} \tag{3.39}$$

Our main result can be summarized in the following.

Theorem Suppose $f(x,y)$ is a sufficiently smooth fuction in $G+\Gamma$, and satisfies condition (1.5). Then 1) The asymptotic expansion (2.1) of the solution of $u(x,y,\varepsilon)$ for problems (1.1), (1.2) uniformly hold in $G+\Gamma$ 2) The numerical solution of (1.1), (1.2) and can be defined by difference problems (3.1) ,(3.2); (3.8), (3.9); (3.15), (3.16); (3.22), (3.23); (3.28), (3.29) and (3.34), (3.35), and the error estimate (3.39) holds.

Remark 1 The above results can be extended without difficulty to the more general equation:

$$\mathscr{L}_\varepsilon u\equiv\varepsilon^2\Delta u-A(x,y)\frac{\partial u}{\partial y}-k^2(x,y)u=f(x,y)$$

where $A(x,y)>0,\ k(x,y)>0,\ f(0,0)=0,\ f(a,0)=0$

Remark 2 If $A(x,y)<0$, then $f(0,b)=0, f(a,b)=0$ will be required.

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