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FIXED POINT INDEX OF ULTIMATELY COMPACT SET-VALUED MAPPINGS IN HAUSDORFF LOCALLY CONVEX SPACES AND ITS APPLICATIONS*

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Abstract

The author defines a concept of fixed point index of ultimately compact set-valued mappings in Hausdorff locally convex spaces. Using this concept, the author establishes several nonzero fixed point theorems of set-valued Φ -condensing mappings. These theorems extend some known results in [1,2,7,8,9].

I. Introduction

Fitzpatrick and petryshyn[1] defined the concept of fixed point index of condensing set-valued mappings in Frechet spaces. Using the concept, they proved some existence theorems of nonzero fixed point of the mappings. Petryshyn and Fitzpatrick[2] established the theory of topological degree of ultimately compact set-valued mappings in a locally convex space in which each convex subset is supposed to be a retract. As a generalization of the above mentioned concept, Duc, Thanh and Ang[3] defined the concept of topological degree of ultimately compact set-valued vector fields in not necessarily metrizable topological vector spaces.

In this paper, first we establish a concept of fixed point index of ultimately compact set-valued mappings on closed and convex subset in general Hausdorff locally convex spaces. Then using the concept, we prove several existence theorems of nonzero fixed point of set-valued Φ -condensing mappings in general Hausdorff locally convex spaces. our theorems generalize and improve many known results in [1,2,7,8,9].

II. Fixed Point Index of Set-valued Ultimately Compact Mappings

Let X be a Hausdorff locally convex space and the topology on X is determined by the family of seminorms $\{p_a;\ a\in A\}$. $K(X)$ denotes the family of all nonempty closed convex subsets of X.

A mapping T: $D\subseteq X\to K(X)$ is called upper semi-continuous (u.s.c.) if for each $x\in D$ and open set $V\subset X$ with $T(x)\subset V$, there exists an open set $W\subset X$ with $x\in W$ such that $T(W\cap D)\subseteq V$.

Now let F be a closed convex subset of X and $\Omega\subseteq X$ be an open subset with $\Omega_F=\Omega\cap F\neq\phi$. We denote by $\overline{\Omega}_F$ and $\partial(\Omega_F)$ the closure and the boundary, respectively, of Ω_F with respect to F.

Suppose T: $\overline{\Omega}_F\to K(F)$ be a u.s.c. mapping. We define a transfinite sequence $\{K_a\}$ by

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induction. Let $K_0=\overline{\text{co}}\, T(\overline{\Omega}_F)$. Suppose α is an ordinal such that K_β has been defined for all ordinals $\beta<\alpha$. Then, if α is an ordinal of first kind, we let $K_\alpha=\overline{\text{co}}\, T\ (K_{\alpha-1}\cap\overline{\Omega}_F)$, while if α is an ordinal of the second kind we let $K_\alpha=\bigcap_{\beta<\alpha}K_\beta$. Then, as in [2], we can show that each K_α is closed and convex with $K_\alpha\subseteq K_\beta$ for $\alpha\geqslant\beta$; $T(K_\alpha\cap\overline{\Omega}_F)\subseteq K_\alpha$ for each ordinal α . Since the transfinite sequence $\{K_\alpha\}$ is nonincreasing, there is an ordinal γ such that $K_\eta=K_\gamma$ for $\eta\geqslant\gamma$, $\{x\,|\,x\in\overline{\Omega}_F,\ x\in T(x)\}\subseteq K_\gamma$ and $\overline{\text{co}}T(K_\gamma\cap\overline{\Omega}_F)=K_\gamma$. We write $K=K_\gamma=K(T,\overline{\Omega}_F)$.

Definition 1 A u.s.c. mapping T: $\overline{\Omega}_F\to K(F)$ is called ultimately compact if either $K\cap\overline{\Omega}_F=\phi$, where $K=K(T,\ \overline{\Omega}_F)$, or if $K\cap\overline{\Omega}_F\neq\phi$, then $T(K\cap\overline{\Omega}_F)$ is a relatively compact set.

Definition 2 Let T: $\overline{\Omega}_F\to K(F)$ be an ultimately compact mapping such that $x\notin T(x)$, $\forall x\in\partial(\Omega_F)$. By the definition of $K=K(T,\ \overline{\Omega}_F)$, K has all properties of the K in the Definition 1 of [3]. Hence, from the argument in [3], it follows that the relatively topological degree $\deg_K(I\text{-}T,\ \Omega,\theta)$ is well defined where I is the identity mapping. Define

$$i_F(T,\Omega)=\deg_K(I\text{-}T,\Omega,\theta)$$

$i_F(T,\Omega)$ is said to be the fixed point index of T over Ω with respect to F.

From the results in [3], we see that the index $i_F(T,\Omega)$ has the following properties.

Theorem 1 Let T: $\overline{\Omega}_F\to K(F)$ be an ultimately compact mapping such that $x\notin T(x)$ for $x\in\partial(\Omega_F)$. Then

(P_1) if $i_F(T,\Omega)\neq 0$, then there exists an $x\in\Omega_F$ such that $x\in T(x)$,

(P_2) if $x_0\in\Omega_F$, then $i_F(\hat{x}_0,\ \Omega)=1$, where $\hat{x}_0$ denotes the map whose constant value is $\{x_0\}$.

(P_3) if $\Omega_1,\ \Omega_2$ is a pair of disjoint open subsets of Ω such that $x\notin T(x),\forall x\in(\overline{\Omega}\backslash(\Omega_1\cup\Omega_2))_F=F\cap(\overline{\Omega}\backslash(\Omega_1\cup\Omega_2))$, then $i_F(T,\Omega)=i_F(T,\Omega_1)+i_F(T,\Omega_2)$, .

(P_4) if $H:[0,1]\times\overline{\Omega}_F\to K(F)$ is an ultimately compact set-valued map such that $x\notin H(t,x),\forall(t,x)\in[0,1]\times\partial(\Omega_F)$, then $i_F(H(1,\cdot),\ \Omega)=i_F(H(0,\cdot)\Omega)$.

Proof Since $K=K(T,\overline{\Omega}_F)\subseteq F$, $K\cap\Omega\subseteq F\cap\Omega=\Omega_F$ and $K\cap\partial\Omega\subseteq F\cap\partial\Omega=\partial(\Omega_F)$, from the argument of Theorems 6 and 7 of [3], it follows that the conclusions (P_1), (P_3) and (P_4) hold. Noting that $\hat{x}_0(x)=\{x_0\}$, $\forall x\in\overline{\Omega}_F$ and $x_0\in\Omega_F$, from the definition of $K=K(\hat{x}_0,\ \overline{\Omega}_F)$ it follows that $x_0\in K\cap\Omega$. Hence (P_2) holds easily.

III. Nonzero Fixed Points of Set-valued Condensing Mappings

In this section, using the fixed point index of ultimately compact set-valued mappings, we shall consider the nonzero fixed point problem of set-valued condensing mappings.

Let X be a Hausdorff locally convex space and its topology is determined by the family of seminorms $\{p_\alpha:\ \alpha\in A\}$. Given $\alpha\in A$ and $\Omega\subseteq X$, we define

$\chi_\alpha(\Omega)$ $=$ $\inf\{\varepsilon>0|$ there exists $x_i\in X$, $i=1,\dots,n$ such that $\Omega\subseteq\bigcup_{i=1}^{n}B_\alpha(x_i,\varepsilon)\}$ with $B_\alpha(x_i,\varepsilon)=\{y\in X\,|\ p_\alpha(x_i-y)<\varepsilon\}$,

and

$\gamma_\alpha(\Omega)=\inf\{d>0\,|\,\Omega$ can be contained in the union of a finite number of sets, each of which has p_α-diameter less that $d\}$.

Now we let $C=\{\phi:\ A\to R^+=[0,\infty)\}$. For $\phi,\ \psi\in C$ and $\lambda>0$, we define

$$\phi\leqslant\psi\Leftrightarrow\phi(\alpha)\leqslant\psi(\alpha)\qquad(\forall\alpha\in A);$$

$(\lambda\phi)(\alpha)=\lambda\phi(\alpha) \quad (\forall\alpha\in A);$

$0(\alpha)=0 \quad (\forall\alpha\in A)$

$(\max\{\phi,\psi\})(\alpha)=\max\{\phi(\alpha),\ \psi(\alpha)\},\ (\forall\alpha\in).$

We now define the mappings χ: $2^X\to C$ and γ: $2^X\to C$ as follows:

$$\chi(\Omega)(\alpha)=\chi_\alpha(\Omega), \quad (\forall\alpha\in A),$$
$$\gamma(\Omega)(\alpha)=\gamma_\alpha(\Omega), \quad (\forall\alpha\in A).$$

The mappings χ and γ are called the χ-measure and γ-measure of non-compactness. The χ-measure of noncompactness was first introduced by Sadovskii [4] and the γ-measure of noncompactness was introduced by Petryshyn, Fitzpatrick [2]. For the properties of χ and γ, the reader may consult [2].

We recall that a closed convex subset W of Hausdorff locally convex space X is called a wedge if $tx\in W$ for each $x\in W$ and $t\in R^+$. If the wedge W satisfies $W\cap\{-W\}=\{\theta\}$, then we say that W is a cone in X. In the following, we denote the cone by P.

In the following we always assume $\Phi=\chi$ or γ.

Definition 3 A u.s.c. mapping T: $D\subseteq X\to K(X)$ is called Φ-condensing if for each $\Omega\subseteq D$ which is not relatively compact there exists $\alpha\in A$ such that $\Phi(T(\Omega))(\alpha)<\Phi(\Omega)(\alpha)$

Now we denote the family of all compact convex subsets of X by $CK(X)$.

Lemma 1 Let T: $\overline{\Omega}_P\to CK(X)$ be a Φ-condensing mapping. Then T is ultimately compact and hence the fixed point index of T has the properties in Theorem 1.

Proof By the construction of $K=K(T,\ \overline{\Omega}_P)$, $\overline{\text{co}}T(K\cap\overline{\Omega}_P)=K$. If $K\cap\overline{\Omega}_P=\phi$, then the Lemma holds. Suppose $K\cap\overline{\Omega}_P\neq\phi$. We have that

$$\Phi(T(K\cap\overline{\Omega}_P))(\alpha)=\Phi(\overline{\text{co}}\,T(K\cap\overline{\Omega}_P))(\alpha)=\Phi(K)(\alpha)$$
$$\geqslant\Phi(K\cap\overline{\Omega}_P)(\alpha) \qquad (\forall\alpha\in A)$$

Since T is condensing, it follows from the above inequality that $K\cap\overline{\Omega}_P$ is compact. Since T is a u.s.c. mapping with compact value, therefore $T(K\cap\overline{\Omega}_P)$ is compact and so T is ultimately compact.

Theorem 2 Let X be a Hausdorff locally convex space and $\Omega\subsetneq X$ be an open set containing θ Suppose T: $\overline{\Omega}_W\to CK(W)$ is a Φ-condensing mapping such that

$$\lambda x\notin T(x),\ \forall x\in\partial(\Omega_W) \qquad (\lambda\geqslant1) \tag{3.1}$$

Then $i_W(T,\Omega)=1$ and T has a fixed point in Ω_W.

Proof Define the mapping H: $[0,1]\times\overline{\Omega}_W\to CK(W)$ by

$$H(t,x)=tT(x),\ \forall\ (t,x)\in[0,1]\times\overline{\Omega}_W$$

If $Q\subseteq\overline{\Omega}_W$ is not relatively compact, then there exists $\alpha\in A$ such that

$$\Phi(T(Q))(\alpha)<\Phi(Q)(\alpha)$$

Since $H([0,1]\times Q)\subseteq\overline{\text{co}}(T(Q)\cup\{\theta\})$, we have

$$\Phi(H([0,1]\times Q))(\alpha)\leqslant\Phi(\overline{\text{co}}(T(Q)\cup\{\theta\}))(\alpha)$$
$$\leqslant\Phi(T(Q))(\alpha)<\Phi(Q)(\alpha)$$

Thus H is Φ -condensing. From Lemma 1 it follows that T is ultimately compact. Since (3.1) implies $x\notin H(t,x)$, $\forall(t,x)\in[0,1]\times\partial(\Omega_W)$. By ($P_4$) and ($P_2$) of Theorem 1 we have

$$i_W(T,\Omega)=i_W(\hat{\theta},\ \Omega)=1$$

By (P_1) of Theorem 1, T has a fixed point in Ω_W.

Remark 1 Note that X is general Hausdorff locally convex space without the retraction condition in [1,2]. Putting $\overline{W}=X$, we obtain the improvement of Theorem 3.2 of [2]. Theorem 3.1 of [1] is also a special case of our Theorem 2.

Lemma 2 Let Ω be a bounded open subset of a Hausdorff locally convex space X. T: $\overline{\Omega}_W\to CK(W)$ is a Φ-condensing set-valued mapping and B: $\overline{\Omega}_W\to CK(W)$ is a compact set-valued mapping. If the following conditions are satisfied:

(i) for each $\alpha\in A$, the set $(I-T)(\overline{\Omega}_W)$ is p_α-bounded and

$$\inf_{x\in\overline{\Omega}_W}\inf_{y\in Bx}p_\alpha(y)>0$$

(ii) $x\notin T(x)+tB(x)$, $\forall(t,x)\in[0,\infty)\times\partial(\Omega_W)$

Then $i_W(T,\Omega)=0$

Proof Suppose $i_W(T,\Omega)\neq0$. For any $\lambda>0$, define the mapping H: $[0,1]\times\overline{\Omega}_W\to CK(W)$ by

$$H(t,x)=T(x)+t\lambda B(x)$$

By (ii), we have $x\notin H(t,x)$, $\forall(t,x)\in[0,1]\times\partial(\Omega_W)$

Now if $Q\subseteq\overline{\Omega}_W$ is not relatively compact, there exists $\alpha\in A$ such that

$$\Phi(T(Q))(\alpha)<\Phi(Q)(\alpha)$$

Since $H([0,1]\times Q)\subseteq T(Q)+\overline{\text{co}}(\lambda B(x)\cup\{\theta\})$ and $\Phi(\lambda B(Q)\cup\{\theta\})(\alpha)=\lambda\Phi(B(Q))(\alpha)=0$, we have

$$\begin{aligned}\Phi(H([0,1]\times Q))(\alpha)&\leqslant\Phi(T(Q))(\alpha)+\Phi(\overline{\text{co}}(\lambda B(Q)\cup\{\theta\}))(\alpha)\\&\leqslant\Phi(T(Q))(\alpha)<\Phi(Q)(\alpha)\end{aligned}$$

It follows from Lemma 1 that H is ultimately compact. By (P_4) of Theorem 1, $i_W(T+\lambda B,\Omega)=i_W(T,\Omega)\neq0$. From ($P_1$) of Theorem 1 it follows that for each $\lambda>0$ there exists $x_\lambda\in\Omega_W$ such that $x_\lambda\in T(x_\lambda)+\lambda B(x_\lambda)$. Since $\inf_{\lambda>0}\inf_{y\in B(x_\lambda)}p_\alpha(y)>0$, it follows that $(I-T)(\overline{\Omega}_W)$ is a unbounded set. This is in contradiction with the condition (i). Hence $i_W(T,\Omega)=0$

Lemma 3 Let P be a cone of a locally convex metrizable space X and $\Omega\subseteq X$ be a bounded open set. Suppose T: $\overline{\Omega}_P\to CK(P)$ is Φ-condensing and B: $\partial(\Omega_P)\to CK(P)$ is a compact set-valued mapping such that

(i) $\inf_{x\in\partial(\Omega_P)}\inf_{y\in B(x)}|y|>0$, where $|\cdot|$ is a quasinorm on X and the distance $d(x,y)=|x-y|$ generates the topology of X. (See. [5], 1.6.1)

(ii) $x\notin T(x)+tB(x)$, $\forall x\in\partial(\Omega_P)$, $t\geqslant0$

Then $i_P(T,\Omega)=0$

Proof Since $\partial(\Omega_P)=P\cap\partial\Omega$ is a bounded closed set in X, by Theorem 1 of [6], B has a u.s.c. compact set-valued extension (still denote it by B) B: $\overline{\Omega}_P\to CK(P)$ such that

$$B(\overline{\Omega}_P)\subseteq\overline{\text{co}}B(\partial(\Omega_P))\tag{3.2}$$

Since $B(\partial(\Omega_P))$ is relative compact, the condition (i) implies $0\notin\overline{B(\partial(\Omega_P))}$. It follows easily that

$$\inf_{z\in\overline{co}B(\partial(\Omega_P))} |z|>0 \tag{3,3}$$

By (3.2) and (3.3), we have

$$\inf_{x\in\overline{\Omega}_P}\ \inf_{y\in B(x)} |y|>0$$

Since $\overline{\Omega}_P$ is a bounded closed set, it is easy to show that $T(\overline{\Omega}_P)$ is a bounded set and so $(I-T)(\overline{\Omega}_P)$ is a bounded set. The conclusion of Lemma 3 holds from Lemma 2.

Remark 2 Lemma 3 is the improvement and generalization of Lemma 1 of [7].

Lemma 4 Let P be a cone of a locally convex metrizable space X and $\Omega\subseteq X$ be a bounded open set. Suppose T: $\overline{\Omega}_P\to CK(P)$ is a compact set-valued mapping such that

(i)′ $\inf_{x\in\partial(\Omega_P)}\ \inf_{y\in T(x)} |y|>0$

(ii)′ $\mu x\in T(x),\ x\in\partial(\Omega_P)\Rightarrow\mu\notin(0,1]$

Then $i_P(T,\Omega)=0$

Proof Letting $T=B$ in Lemma 3, we show that the conditions of Lemma 3 hold. obviously, T is Φ-condensing. We only need to prove (ii)′ $\Rightarrow$ (ii) of Lemma 3. If it is not true, then there is an $x_0\in\partial(\Omega_P)$ and $t_0\geqslant 0$ such that

$$x_0\in(1+t_0)T(x_0)$$

Thus we have $0<\frac{1}{1+t_0}\leqslant 1$ and $\frac{1}{1+t_0}x_0\in T(x_0)$. This is in contradition iction with (ii)′. By Lemma 3, this lemma holds.

Remark 3 Lemma 4 extends Lemma 2 of [7].

Theorem 3 Let Ω_1 and Ω_2 be bounded open subset of a Hausdorff locally convex space X with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and T: $\overline{\Omega}_{2,W}\to CK(W)$ is Φ-condensing. If one of the following conditions holds:

(H₁) $\begin{cases}\text{for each } \alpha\in A \text{, the set } (I-T)(\overline{\Omega}_{2,W}) \text{ is } p_\alpha\text{-bounded and there exists compact set-}\\ \text{valued map } B\text{: } \overline{\Omega}_{2,W}\to CK(W) \text{ such that } \inf_{x\in\overline{\Omega}_{2,W}}\ \inf_{y\in B(x)} p_\alpha(y)>0,\ x\notin T(x)+tB(x),\\ \forall\, x\in\partial(\Omega_{2,W}),\ t>0;\\ \lambda x\notin T(x),\ \forall x\in\partial(\Omega_{1,W}),\ \lambda>1.\end{cases}$

(H₂) $\begin{cases}\text{for each } \alpha\in A \text{, the set } (I-T)(\overline{\Omega}_{1,W}) \text{ is } p_\alpha\text{-bounded and there is a compact set-valued}\\ \text{map } B\text{: } \overline{\Omega}_{1,W}\to CK(W) \text{ such that } \inf_{x\in\overline{\Omega}_{1,W}}\ \inf_{y\in B(x)} p_\alpha(y)>0,\ x\notin T(x)+tB(x),\ \forall x\in\partial\\ (\Omega_{1,W}),\ t>0;\\ \lambda x\notin T(x),\ \forall x\in\partial(\Omega_{2,W}),\ \lambda>1.\end{cases}$

Then T has a fixed point in $\overline{\Omega}_{2,W}\backslash\Omega_{1,W}$.

Proof Suppose that (H_1) holds (when (H_2) holds the proof is similar). If T has a fixed point in $\partial(\Omega_{2,W})\cup\partial(\Omega_{1,W})$, then this theorem holds. Now assume that $x\notin T(x),\ \forall x\in\partial(\Omega_{2,W})\cup\partial(\Omega_{1,W})$. It follows from Theorem 2 and Lemma 2 that

$$i_W(T,\Omega_1)=1 \quad\text{and}\quad i_W(T,\Omega_2)=0$$

Since $x \notin T(x)$ for $x \in W \cap [\Omega_2 \backslash ((\Omega_2 \backslash \overline{\Omega}_1) \cup \Omega_1] = \partial(\Omega_{2,W}) \cup \partial(\Omega_{1,W})$, by ($P_3$) of Theorem 1, we have

$$i_W(T,\Omega_2) = i_W(T,\Omega_2 \backslash \overline{\Omega}_1) + i_W(T,\Omega_1)$$

and so $i_W(T,\Omega_2 \backslash \overline{\Omega}_1) = -1$. By (P_1) of Theorem 1, T has a fixed point in $W \cap (\Omega \backslash \overline{\Omega}_1)$

Corollary 1 Let Ω_1 and Ω_2 be bounded open subsets of a Husdorff locally convex space X with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $T: \overline{\Omega}_{2,W} \to CK(W)$ is Φ -condensing. If one of the following conditions holds:

$$(H_1) \begin{cases} \text{for each } \alpha \in A, \ (I-T)(\overline{\Omega}_{2,W}) \text{ is } p_\alpha\text{-bounded and there exists a } h \in W \text{ with} \\ p_\alpha(h) \neq 0 \text{ such that } x \notin T(x) + th \text{ for } x \in \partial(\Omega_{2,W}) \text{ and } t > 0; \ \lambda x \notin T(x), \\ \forall x \in \partial(\Omega_{1,W}), \ \lambda > 1. \end{cases}$$

$$(H_2)' \begin{cases} \text{for each } \alpha \in A, \ (I-T)(\Omega_{1,W}) \text{ is } p_\alpha\text{-bounded and there exists a } h \in W \text{ with} \\ p_\alpha(h) \neq 0 \text{ such that } x \notin T(x) + th \text{ for } x \in \partial(\Omega_{1,W}) \text{ and } t > 0; \ \lambda x \notin T(x), \forall x \in \\ \partial(\Omega_{2,W}), \ \lambda > 1. \end{cases}$$

Then T has a fixed point in $\overline{\Omega}_{2,W} \backslash \Omega_{1,W}$

Proof Putting $B(x) = \{h\}$, $\forall x \in \overline{\Omega}_{2,W}$ in Theorem 3, we obtain corollary 1.

Remark 4 Corollary 1 is the improvement and generalization of Theorem 3 of [1].

Theorem 4 Let P be a cone of a Locally convex metrizable space X. Ω_1 and Ω_2 are bounded open subsets of X with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, T: $\Omega_{2,P} \to CK(P)$ is Φ -condensing. If one of the following conditions holds:

$$(H_3) \begin{cases} \text{There is a compact set-valued map } B: \partial(\Omega_{1,P}) \to CK(P) \text{ such that } \inf\limits_{x \in \partial(\Omega_{1,P})} \inf\limits_{y \in B(x)} |y| \\ > 0, \ x \notin T(x) + tB(x), \ \forall x \in \partial(\Omega_{1,P}), \ t > 0; \\ \lambda x \notin T(x), \ \forall x \in \partial(\Omega_{2,P}), \ \lambda > 1. \end{cases}$$

$$(H_4) \begin{cases} \text{there exists a compact set-valued map } B: \partial(\Omega_{2,P}) \to CK(P) \text{ such that } \inf\limits_{x \in \partial(\Omega_{2,P})} \inf\limits_{y \in B(x)} \\ |y| > 0, \ x \notin T(x) + tB(x), \ \forall x \in \partial(\Omega_{2,P}), t > 0; \\ \lambda x \notin T(x), \ \forall x \in \partial(\Omega_{1,P}), \ \lambda > 1. \end{cases}$$

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Proof The proof is similar to that of Theorem 3, so we omit it.

Theorem 5 Let P, X, Ω_1 and Ω_2 be the same as in Theorem 4. $T: P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to CK(P)$ is a compact set-valued map. If one the following conditions holds:

$$(H_5) \begin{cases} \inf\limits_{x \in \partial(\Omega_{1,P})} \inf\limits_{y \in T(x)} |y| > 0, \ \mu x \in T(x), \ x \in \partial(\Omega_{1,P}) \Rightarrow \mu \geqslant 1; \\ \mu x \in T(x), \ x \in \partial(\Omega_{2,P}) \Rightarrow \mu \leqslant 1. \end{cases}$$

$$(H_6) \begin{cases} \inf\limits_{x \in \partial(\Omega_{2,P})} \inf\limits_{y \in T(x)} |y| > 0, \ \mu x \in T(x), \ x \in \partial(\Omega_{2,P}) \Rightarrow \mu \geqslant 1; \\ \mu x \in T(x), \ x \in \partial(\Omega_{1,P}) \Rightarrow \mu \leqslant 1. \end{cases}$$

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$

Proof Suppose (H_5) holds (when (H_6) holds the proof is similar). It T has a fixed point in $\partial(\Omega_{1,P}) \cup \partial(\Omega_{2,P})$, then the theorem holds. Now assume that T has not fixed point in $\partial(\Omega_{1,P}) \cup \partial(\Omega_{2,P})$. By Theorem 2.1 of [6], B has a u.s.c. compact set-valued extension (still denote it by B) B: $\overline{\Omega}_{2,P} \to CK(P)$. From (H_5) and Lemma 4 it follows that $i_P(T,\Omega_1) = 0$. It is easy to see that

(H_5) implies (3.1) with $W=P$. By Theorem 2, we have $i_P(T,\Omega_2)=1$. The remainder of proof is the same as that of Theorem 3.

Remark 5 Theorem 5 extends Theorem 1 of [7], Corollary 3.3 of [1] and Theorems 1.2 and 1.3 of [8].

Now suppose that $B: P\to CK(P)$ is a compact set-valued map. Let $P_B=\{x\in P\mid \exists\lambda>0,\ y\in B(x)$ such that $x-\lambda y\in P\}$. Specially, if $B(x)\equiv\{h\}$ with $0\neq h\in P$ for $x\in P$, we write $P_h=\{x\in P\mid \exists\lambda>0$ such that $x-\lambda h\in P\}$

Theorem 6 Let Ω_1 and Ω_2 be bounded open subsets of a Hausdorff locally convex space X with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$. T: $\overline{\Omega}_{2,P}\to CK(P)$ is Φ -condensing. If one of the following conditions holds:

$$(H_7)\begin{cases}\text{for each } \alpha\in A,\ (I-T)\ (\overline{\Omega}_{2,P}) \text{ is } p_\alpha\text{-bounded and } \inf\limits_{x\in\Omega_{2,P}}\inf\limits_{y\in B(x)} p_\alpha(x)>0 \text{ and}\\ y\nleq x,\ \ \forall x\in P_B\cap\partial\Omega_2,\ \ y\in T(x);\\ y\ngeq(1+\varepsilon)x,\ \ \forall\, x\in P\cap\partial\Omega_1,\ \ y\in T(x) \text{ and } \varepsilon>0\,.\end{cases}$$

$$(H_8)\begin{cases}\text{for each } \alpha\in A,\ (I-T)(\overline{\Omega}_{1,P}) \text{ is } p_\alpha\text{-bounded and } \inf\limits_{x\in\overline{\Omega}_{1,P}}\inf\limits_{y\in B(x)} p_\alpha(y)>0 \text{ and}\\ y\nleq x,\ \ \forall x\in P_B\cap\partial\Omega_1,\ \ y\in T(x);\\ y\ngeq(1+\varepsilon)x, \forall\, x\in P\cap\partial\Omega_2,\ \ y\in T(x) \text{ and } \varepsilon>0\,.\end{cases}$$

Then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Proof Suppose (H_7) holds (when (H_8) holds, the proof is similar). We show that (H_7) $\Rightarrow$ (H_1) with $W=P$. If the condition $x\notin T(x)+tB(x),\ \forall x\in\partial(\Omega_{2,P}), t>0$ is not true, then there exist $x_0\in\partial(\Omega_{2,P}),\ t_0>0,\ y_0\in T(x_0)$ and $z_0\in B(x_0)$ such that $x_0=y_0+t_0 z_0$.Hence $x_0-t\,z_0=y_0\in T(x_0)\subseteq P$and so $x_0\in P_B\cap\partial\Omega_2$. Since $x_0-y_0=t_0 z_0\in t_0 B(x_0)\subseteq P$, we have $y_0\leqslant x_0$. This is in contradiction with (H_7). If the condition $\lambda x\notin T(x),\ \forall x\in\partial(\Omega_{1,P}),\ \lambda>1$ is not true, then there exist $x_0\in\partial(\Omega_{1,P}),\ \lambda_0>1$ and $y_0\in T(x_0)$ such that $\lambda\, x_0=y_0$. Letting $0<\varepsilon_0<\lambda_0-1$, we have $1+\varepsilon_0<\lambda_0$ and $y_0\geqslant(1+\varepsilon_0)x_0$. This is in contradiction with (H_7). From Theorem 3 with $W=P$, it follows that Theorem 6 holds.

Remark 6 Theorem 6 is the generalization of Corollary 3.1 of [1].

Theorem 7 Let P, X, Ω_1 and Ω_2 be the same as in Theorem 4. T: $\overline{\Omega}_{2,P}\to CK(P)$ is Φ-condensing. If one of the following conditions holds:

$$(H_9)\begin{cases}\inf\limits_{x\in\partial(\Omega_{1,P})}\inf\limits_{y\in B(x)}|y|>0,\ \ y\nleq x,\ \ \forall x\in P_B\cap\partial\Omega_1,\ \ y\in T(x);\\ y\ngeq(1+\varepsilon)x,\ \ \forall\, x\in P\cap\partial\Omega_2,\ \ y\in T(x) \text{ and } \varepsilon>0\,.\end{cases}$$

$$(H_{10})\begin{cases}\inf\limits_{x\in\partial(\Omega_{2,P})}\inf\limits_{y\in B(x)}|y|>0,\ \ y\nleq x,\ \ \forall x\in P_B\cap\partial\Omega_2,\ \ y\in T(x);\\ y\ngeq(1+\varepsilon)x,\ \ \forall x\in P\cap\partial\Omega_1,\ \ y\in T(x) \text{ and} \varepsilon>0\,.\end{cases}$$

Then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Proof Using similar argument as in Theorem 6, we easily show that (H_9) $\Rightarrow$ (H_3) and (H_{10}) $\Rightarrow$(H_4). This theorem follows from Theorem 4.

Remark 7 Theorem 7 improves and extends Theorem 2 of [7].

Corollary 2 Let P, X, Ω_1 and Ω_2 be the same as in Theorem 4 T: $\overline{\Omega}_{2,P}\to CK(P)$ is Φ-condensing. If one of the following conditions holds:

$$(\mathrm{H}_9)'\begin{cases} y\nleq x, & \forall x\in P_h\cap\partial\Omega_1,\quad y\in T(x), \\ y\ngeq(1+\varepsilon)x, & \forall x\in P\cap\partial\Omega_2,\quad y\in T(x)\text{ and }\varepsilon>0, \end{cases}$$

$$(\mathrm{H}_{10})'\begin{cases} y\nleq x, & \forall x\in P_h\cap\partial\Omega_2,\quad y\in T(x) \\ y\ngeq(1+\varepsilon)x, & \forall x\in P\cap\partial\Omega_1,\quad y\in T(x)\text{ and }\varepsilon>0. \end{cases}$$

Then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Proof By putting $B(x)=\{h\},\ \forall x\in P$, where $h\in P,\ |h|\neq 0$, this corollary follows from Theorem 7.

Remark 8 Corollary 2 extends the corollary of [7] and Theorem 2 of [9].

References

[1] Fizpatrick, P.M. and W.V. Petryshyn, Fixed point theorems and the fixed point index for multivalued mappings in cones, *J. London Math. Soc.*, 12 (1975), 75 – 85.

[2] Petryshyn, W.V. and P.M. Fizpatrick, A degree theory, fixed point theroems, and mapping theorem for multivalued noncompact mappings, *Trans. Amer. Math. Soc.*, 194 (1974), 1 – 25.

[3] Duc, D.M. D.D. Thanh and D.D. Ang, Relative topological degree of set-valued compact vector fields and its applications, *J. Math. Anal. appl.*, 80 (1981), 406 – 432.

[4] Sadovskii, B.N., Measures of noncompactness and condensing operators, *Prob. Mat. Anal. Sloz Sistem*, 2 (1968), 89 – 119.

[5] Schaefer, H.H., *Topological Vector Spaces*, Springer-Verlag, New York, Fourth, Corrected Printing (1980).

[6] Ma, T.W., Topological degrees for set-valued compact vector fields in locally convex spaces, *Dissertationes Math.*, 92 (1972), 1 – 43.

[7] Guo Da-jun, *Kexue Tongbao*, **28**, 20 (1983), 1217 – 1219. (in Chinese)

[8] Gatica, J.A. and H.L. Smith, Fixed point techniques in cone with applications, *J. Math. Anal. Appl.*, 61 (1977), 58 – 71.

[9] Leggett, R.W. and L.A. Williams, A fixed point theorem with application to an infectious disease model, *J. Math. Anal. Appl.*, 76 (1980), 91 – 97.