Applied Mathematics and Mechanics
(English Edition, Vol.8, No.3, Mar. 1987)

Published by SUT,
Shanghai, China

SOME EXTENSIONS OF THE INITIAL VALUE PROBLEM FOR SECOND ORDER EVOLUTION EQUATIONS

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(Received April 19, 1985 Communicated by Guo You-zhong)

Abstract

The initial problem for second order linear evolution equation systems is discussed by using the contraction semigroup theory. A kind of initial value problem for second order is also discussed with variable coefficients for evolution equations by using the analytical semigroup theory, and is unified with the solutions of the initial value problem for this class of equations and those of first order temporally inhomogeneous evolution equations. This is an important class of equations in mathematical mechanics.

I. Introduction

Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume $\mathscr{A}\in\mathscr{L}(V,V')$ and $\mathscr{C}\in\mathscr{L}(W,W')$ are the Riesz maps of V and W respectively, and let B be linear from the subspace $D(B)$ of V into V'. For every $f\in C^1(R^+,W')$ [1] discussed the initial and initial-boundary value problems for

$$\mathscr{C}u''(t)+Bu'(t)+\mathscr{A}u(t)=f(t) \tag{1.1}$$

[3] made (1.1) an extension to

$$\left.\begin{aligned}&\mathscr{C}_1u''(t)+\mathscr{B}v'(t)+\mathscr{A}_1u(t)=f_1(t)\\&\mathscr{C}_2v''(t)-\mathscr{B}u'(t)+\mathscr{A}_2v(t)=f_2(t)\end{aligned}\quad (t\geqslant 0)\right\} \tag{1.2}$$

where $\mathscr{A}_ix(y)=a_i(x,y)$, $(x,y\in V)$, $\mathscr{C}_ix(y)=c_i(x,y)$, $(x,y\in W)$ and $a_i(\cdot,\cdot)$, $c_i(\cdot,\cdot)$ are sesquilinear symmetric continuous elliptic forms, $(i=1,2)$ $[f_1(\cdot),\ f_2(\cdot)]\in C'(R_0^+,W'\times W')$.

This paper will discuss the Cauchy's problem for

$$\left.\begin{aligned}&\mathscr{C}_{11}u''(t)+\mathscr{C}_{12}v''(t)+\mathscr{B}_{11}u'(t)+\mathscr{B}_{12}v'(t)+\mathscr{A}_{11}u(t)+\mathscr{A}_{12}v(t)=f_1(t)\\&\mathscr{C}_{21}u''(t)+\mathscr{C}_{22}v''(t)+\mathscr{B}_{21}u'(t)+\mathscr{B}_{22}v'(t)+\mathscr{A}_{21}u(t)+\mathscr{A}_{22}v(t)=f_2(t)\end{aligned}\quad (t\geqslant 0)\right\} \tag{1.3}$$

and will give examples of the initial-boundary value problem. Therefore the results in [3] are extended.

Spaces and operators are given as follows.

Let V_i and W_i be Hilbert spaces with V_i dense and continuously imbedded in W_i $(i=1,2)$. Assume $\mathscr{A}_{ii}\in\mathscr{L}(V_i,V_i')$ and $\mathscr{C}_{ii}\in\mathscr{L}(W_i,W_i')$ are Riesz maps of V_i and W_i respectively, and

$\mathscr{A}_{12}\in\mathscr{L}(V_2,V_1')$, $\mathscr{A}_{21}\in\mathscr{L}(V_1,V_2')$, $\mathscr{C}_{12}\in\mathscr{L}(W_2, W_1')$, $\mathscr{C}_{21}\in\mathscr{L}(W_1, W_2')$, $D(\mathscr{B}_{11})\leqslant V_1$, $D(\mathscr{B}_{22})\leqslant V_2$, $D(\mathscr{B}_{12})\leqslant V_2$, $D(\mathscr{B}_{21})\leqslant V_1$, $\mathscr{B}_{11}\in L(D(\mathscr{B}_{11}), V_1')$, $\mathscr{B}_{12}\in L(D(\mathscr{B}_{12}), V_1')$, $\mathscr{B}_{21}\in L(D(\mathscr{B}_{21}), V_2')$, $\mathscr{B}_{22}\in L(D(\mathscr{B}_{22}), V_2')$.

Moreover, we will discuss Cauchy's problem

$$\left.\begin{array}{l}\mathscr{C}u''(t)+B(t)u'(t)+\mathscr{A}(t)u(t)=f(t)\\ u(0)=u_0,\ u'(0)=u_1\end{array}\right\}\quad (t\geqslant 0) \tag{1.4}$$

where $\mathscr{A}(t)$ and $B(t)$ are given as follows:

$$\mathscr{A}(t)u(v)\triangleq a(t,u,v),\ B(t)u(v)\triangleq b(t,u,v)\qquad (u,v\in V) \tag{1.5}$$

$a(t,.,.)$ and $b(t,.,.)$ are sesquilinear continuous V-eilliptic forms.

We will unify the solutions of the initial value problem (1.4) and those of first order evolution equations.

II. The Cauchy Problem for Equation (1.3)

Define $V=V_1\times V_2$, $W=W_1\times W_2$,

$$([x_1, x_2], [y_1, y_2])_V=(x_1,y_1)_{V_1}+(x_2,y_2)_{V_2},\quad ([x_1,x_2], [y_1,y_2]\in V)$$
$$([x_1,x_2],[y_1,y_2])_W=(x_1,y_1)_{W_1}+(x_2,y_2)_{W_2},\quad ([x_1,x_2],[y_1,y_2]\in W)$$

then V and W are Hilbert spaces, $V'=V_1'\times V_2'$, $W'=W_1'\times W_2'$, $V\hookrightarrow W$ is continuously.

Define $\mathscr{A}\in\mathscr{L}(V,V')$, $\mathscr{C}\in\mathscr{L}(W,W')$ by

$$\mathscr{A}[x_1,x_2]([y_1,y_2])=\mathscr{A}_{11}x_1(y_1)+\mathscr{A}_{12}x_2(y_1)+\mathscr{A}_{21}x_1(y_2)+\mathscr{A}_{22}x_2(y_2)$$
$$([x_1,x_2], [y_1,y_2]\in V) \tag{2.1}$$
$$\mathscr{C}[x_1, x_2]([y_1, y_2])=\mathscr{C}_{11}x_1(y_1)+\mathscr{C}_{12}x_2(y_1)+\mathscr{C}_{21}x_1(y_2)+\mathscr{C}_{22}x_2(y_2)$$
$$([x_1,x_2], [y_1,y_2]\in W) \tag{2.2}$$

Define $D(B)=(D(\mathscr{B}_{11})\times D(\mathscr{B}_{21}))\times(D(\mathscr{B}_{12})\times D(\mathscr{B}_{22}))$, we have then $D(B)\leqslant V$

Define $B\in 1(D(B), V')$ by

$$B[x_1,x_2]([y_1,y_2])=\mathscr{B}_{11}x_1(y_1)+\mathscr{B}_{12}x_2(y_1)+\mathscr{B}_{21}x_1(y_2)+\mathscr{B}_{22}x_2(y_2)$$
$$([x_1,x_2]\in D(\mathscr{B}), [y_1,y_2]\in V) \tag{2.3}$$

Denote $W=[u,v]$, $f(t)=[f_1(t), f_2(t)]$, from (2.1)$-$(2.3), we have an equation equivalent to (1.3):

$$\mathscr{C}W''(t)+\mathscr{B}W'(t)+\mathscr{A}W(t)=f(t) \tag{2.4}$$

Definition We consider $\mathscr{A}_{12}$ symmetric with $\mathscr{A}_{21}$ if

$$\mathscr{A}_{12}x(y)=\overline{\mathscr{A}_{21}y(x)}$$

Lemma 1 $\mathscr{A}$ defined by (2.1) is symmetric if $\mathscr{A}_{11}$, $\mathscr{A}_{22}$ are symmetric and $\mathscr{A}_{12}$ is symmetric with $\mathscr{A}_{21}$.

Proof Let $\mathscr{A}_{11}$, $\mathscr{A}_{22}$ be symmetric and $\mathscr{A}_{12}$ be symmetric with $\mathscr{A}_{21}$, we have then

$$\mathscr{A}[x_1,x_2]([y_1,y_2])=\overline{\mathscr{A}_{11}y_1(x_1)}+\overline{\mathscr{A}_{21}y_1(x_2)}+\overline{\mathscr{A}_{12}y_2(x_1)}+\overline{\mathscr{A}_{22}y_2(x_2)}$$
$$=\overline{\mathscr{A}[y_1,y_2]([x_1,x_2])}\qquad ([x_1,x_2], [y_1,y_2]\in V) \tag{2.5}$$

If $\mathscr{A}$ is symmetric, let $x_2=y_2=0$, then $\mathscr{A}_{11}$ is symmetric by (2.5). In the same argument,

$\mathscr{A}_{22}$ is symmetric. Let $x_1=y_2=0$, then $\mathscr{A}_{12}$ is symmetric with $\mathscr{A}_{21}$ by (2.5).

Lemma 2 Assume

$$\operatorname{Re}\{\mathscr{A}_{11}x_1(x_1)+\mathscr{A}_{12}x_2(x_1)+\mathscr{A}_{21}x_1(x_2)+\mathscr{A}_{22}x_2(x_2)\}\geqslant\alpha_1\|x_1\|^2_{V_1}+\alpha_2\|x_2\|^2_{V_2} \quad (2.6)$$
$$(\alpha_1,\ \alpha_2>0,\ [x_1,x_2]\in V)$$

or

$$\operatorname{Re}\mathscr{A}_{11}x_1(x_1)\geqslant\beta_1\|x_1\|^2_{V_1}\ \ (\beta_1>0),\ \operatorname{Re}\mathscr{A}_{22}x_2(x_2)\geqslant\beta_2\|x_2\|^2_{V_2}\ \ (\beta_2>0) \quad (2.7)$$

with $\|\mathscr{A}_{12}\|_{\mathscr{L}(V_2,V_1')}+\|\mathscr{A}_{21}\|_{\mathscr{L}(V_1,V_2')}\leqslant 2\beta=2\min(\beta_1,\beta_2)$, then $\mathscr{A}$ is V-elliptic.

Proof If (2.6) holds true, then

$$\operatorname{Re}\{\mathscr{A}[x_1,x_2]([x_1,x_2])\}\geqslant\min(\alpha_1,\alpha_2)[\|x_1\|^2_{V_1}+\|x_2\|^2_{V_2}]=\alpha\|[x_1,x_2]\|^2_V$$
$$(\alpha>0,\ [x_1,x_2]\in V)$$

If (2.7) is right, then by

$$\|\mathscr{A}_{12}x_2(x_1)\|\leqslant\|\mathscr{A}_{12}\|_{\mathscr{L}(V_2,V_1')}\|x_1\|_{V_1}\|x_2\|_{V_2},$$
$$\|\mathscr{A}_{21}x_1(x_2)\|\leqslant\|\mathscr{A}_{21}\|_{\mathscr{L}(V_1,V_2')}\|x_1\|_{V_1}\|x_2\|_{V_2}$$

we have

$$\operatorname{Re}\{\mathscr{A}[x_1,x_2]([x_1,x_2])\}\geqslant\beta\|x_1\|^2_{V_1}+\beta\|x_2\|^2_{V_2}-(\|\mathscr{A}_{12}\|_{\mathscr{L}(V_2,V_1')}$$
$$+\|\mathscr{A}_{21}\|_{\mathscr{L}(V_1,V_2')})\|x_1\|_{V_1}\|x_2\|_{V_2}$$
$$\geqslant\Big[\beta-\frac{1}{2}(\|\mathscr{A}_{12}\|_{\mathscr{L}(V_2,V_1')}+\|\mathscr{A}_{21}\|_{\mathscr{L}(V_1,V_2')})\Big](\|x_1\|^2_{V_1}+\|x_2\|^2_{V_2}$$
$$\geqslant c\|[x_1,x_2]\|^2_V \qquad (c>0,\ [x_1,x_2]\in V)$$

Definition $\mathscr{B}_{12}$ is anti-symmetric with $\mathscr{B}_{21}$ if $\operatorname{Re}(\mathscr{B}_{12}x(y)+\mathscr{B}_{21}y(x))=0$ for $x\in D(\mathscr{B}_{12})$, $y\in D(\mathscr{B}_{21})$.

Lemma 3 Assume $\mathscr{B}_{ii}$ is monotone ($i=1,2$) and $\mathscr{B}_{12}$ is anti-symmetric with $\mathscr{B}_{21}$, then B is monotone.

Proof By

$$\operatorname{Re}\{\mathscr{B}_{11}x_1(x_1)+\mathscr{B}_{12}x_2(x_1)+\mathscr{B}_{21}x_1(x_2)+\mathscr{B}_{22}x_2(x_2)\}$$
$$\geqslant\operatorname{Re}(\mathscr{B}_{11}x_1(x_1)+\mathscr{B}_{22}x_2(x_2))\geqslant 0 \qquad ([x_1,x_2]\in D(B))$$

we have $\operatorname{Re}B[x_1,x_2]([x_1,x_2])\geqslant 0 \quad ([x_1,x_2]\in D(B))$.

By Lemmata 1 – 3 and [1], we have

Theorem 1 Spaces V, W and operators $\mathscr{A}$, B, $\mathscr{C}$ are given as above. Assume $\mathscr{A}$ and $\mathscr{C}$ satisfy conditions in Lemma 1 and Lemma 2, B satisfies conditions in Lemma 3. Let $(\mathscr{A}+B+\mathscr{C}):D(B)\to V'$ is surjective. Then for every $[f_1(t),f_2(t)]\in C^1(R^+_0,W')$ and $[u_0,v_0]\in V$, $[u_1,v_1]\in D(B)$ with $\mathscr{A}[u_0,v_0]+B[u_1,v_1]\in W'$, there exists a unique solution $[u(t),v(t)]\in C(R^+_0,V)\cap C^1(R^+,V)\cap C^1(R^+_0,W)\cap C^2(R^+,W)$ with $[u(0),v(0)]=[u_0,v_0]$, $[u'(0),v'(0)]=[u_1,v_1]$.

III. Examples

Let Ω be bounded and open in R^n and suppose its boundary $\partial\Omega$ is a C^2-manifold of dimension $(n-1)$. Let Γ_1 and Γ_2 be closed subset of $\partial\Omega_1$ with meas $(\Gamma_1)>0$ and meas $(\Gamma_2)>0$.

Let

$$\left.\begin{array}{l} V_1=\{v\in H^1(\Omega): v(s)=0,\ s\in\Gamma_1, \text{a. e.}\} \\ V_2=\{v\in H^1(\Omega): v(s)=0,\ s\in\Gamma_2, \text{a. e.}\} \end{array}\right\} \tag{3.1}$$

It is easy to verify that both V_1 and V_2 are Hilbert spaces.

Let $W_1=W_2=L^2(\Omega)$, $W=W_1\times W_2$. Suppose

$$C_{12}(x)=\overline{C_{21}(x)},\quad \sum_{i,j=1}^{2} C_{ij}(x)\zeta_i\bar{\zeta}_i\geqslant C(|\zeta_1|^2+|\zeta_2|^2)\qquad (x\in\Omega) \tag{3.2}$$

where $C_{ij}(x)\in(\Omega)$, and

$$\mathscr{C}_{ij}u(v)=\int_\Omega C_{ij}(x)u(x)\bar{v}(x)dx\quad (i,j=1,2) \tag{3.3}$$

then $\mathscr{C}$ defined by (2.2), (3.2) and (3.3) is symmetric by Lemma 1 and elliptic by Lemma 2.

Suppose $R(\cdot)\in L^\infty(\Omega)$ and a vector field $\mu(x)=(\mu_1(x),\ \mu_2(x),\ \cdots,\ \mu_N(x))$, $x\in\Omega$, with each $\mu_j(x)\in C^1(\bar{\Omega})$. We define $\mathscr{B}_{11}\in\mathscr{L}(V_1,W_1)$ by

$$\mathscr{B}_{11}u(v)=\int_\Omega\left(R(x)u(x)+\frac{\partial u(x)}{\partial\mu}\right)\bar{v}(x)dx, \tag{3.4}$$

the indicated directional derivative being given by $\dfrac{\partial u(x)}{\partial\mu}=\sum_{j=1}^{N}\partial_j u(x)\mu_j(x)$.

If

$$\operatorname{Re}R(x)-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial\mu_i(x)}{\partial x_j}\geqslant 0\quad (x\in\Omega)$$

$$\mu(s)n(s)\geqslant 0,\ (s\in\partial\Omega-\Gamma\ ) \tag{3.5}$$

then

$$\operatorname{Re}\mathscr{B}_{11}u(u)=\operatorname{Re}\int_\Omega\left(R(x)u(x)+\frac{\partial u(x)}{\partial\mu}\right)\bar{u}(x)dx\geqslant 0$$

Suppose

$$\mathscr{B}_{22}=0;$$

$$\mathscr{B}_{12}v(\varphi)=\int_\Omega\sum_{j=1}^{N}b_j(x)v_{x_j}(x)\bar{\varphi}(x)dx+\int_\Omega b_0(x)v(x)\bar{\varphi}(x)dx,$$

$$\mathscr{B}_{21}u(\psi)=\int_\Omega\sum_{j=1}^{N}\bar{b}_j(x)u_{x_j}(x)\bar{\psi}(x)dx$$

where

$$b_0=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}b_j,\ \ b_i(x)\in C^1(\Omega),\ \ (j=1,2,\cdots,N,)\text{ with}$$

$$\sum_{j=1}^{N} b_j \cos(n, x_j) \,|_{\partial\Omega - \Gamma_1 \cup \Gamma_2} = 0 \tag{3.6}$$

we have

$$\mathscr{B}_{12}v(\varphi) + \overline{\mathscr{B}_{21}\varphi(v)}$$

$$= \int_\Omega \sum_{j=1}^{N} \left(b_j v_{x_j} \bar{\varphi} + \frac{\partial b_j}{\partial x_j} v\bar{\varphi} + b_j v \bar{\varphi}_{x_j} \right) dx = \int_{\partial\Omega} \sum_{j=1}^{N} b_j v \bar{\varphi} \cos(n, x_j) ds$$

Because $v \in V_2$, $\varphi \in V_1$ and (3.6), we see $\mathscr{B}_{12}$ is anti-symmetric with $\mathscr{B}_{21}$.

Thus B is monotone and $B \in \mathscr{L}(V, W)$.

Let $a_{ij}^k \in C^1(\bar{\Omega}), (k=1,2,3,4;\ i, j=1, 2\cdots, N)$. Let $(a_{ij}^1), (a_{ij}^4)$ by Hermitian matrices. And

$$\overline{a_{ij}^2} = \overline{a_{ji}^3} \quad i, j=1, 2\cdots, N$$

$$\mathscr{A}_{11}u(\varphi) \triangleq \int_\Omega \sum_{i,j=1}^{N} a_{ij}^1(x) u_{x_j} \bar{\varphi}_{x_i} dx, \quad \mathscr{A}_{12}v(\varphi) \triangleq \int_\Omega \sum_{i,j=1}^{N} a_{ij}^2(x) v_{x_j} \bar{\varphi}_{x_i} dx$$

$$\mathscr{A}_{21}u(\psi) \triangleq \int_\Omega \sum_{i,j=1}^{N} a_{ij}^3(x) u_{x_j} \bar{\psi}_{x_i} dx, \quad \mathscr{A}_{22}v(\psi) \triangleq \int_\Omega \sum_{i,j=1}^{N} a_{ij}^4 v_{x_j} \bar{\psi}_{x_i} dx$$

thus $\mathscr{A}$ defined by (2.1) is symmetric by Lemma 1. Suppose

$$(t_{kl}) = \begin{pmatrix} (a_{ij}^1) & (a_{ij}^2) \\ (a_{ij}^3) & (a_{ij}^4) \end{pmatrix}$$

and

$$\sum_{k,l=1}^{2N} t_{kl} \zeta_l \bar{\zeta}_k \geqslant \alpha \sum_{k=1}^{2N} |\zeta_k|^2 \ (\alpha > 0\ )$$

Thus $\mathscr{A}$ is V-elliptic by Lemma 2.

Let $F_1(x,t)$, $F_2(x,t) \in L^2(\Omega \times R_0^+)$, with $\frac{\partial}{\partial t} F_1(x,t), \frac{\partial}{\partial t} F_2(x,t) \in L^2(\Omega \times R_0^+)$, then $f_1(t) = F_1(\cdot, t)$, $f_2(t) = F_2(\cdot, t) \in C^1(R_0^+, L^2(\Omega))$.

For every $[u_0, v_0] \in V \cap (H^2(\Omega) \times H^2(\Omega))$, $[u_1, v_1] \in V$, because $B \in \mathscr{L}(V, W)$, we need $[u_0, v_0] \in V \cap (H^2(\Omega) \times H^2(\Omega))$, $\mathscr{A}[u_0, v_0] \in W$ only. By the definition, we assume

$$\left.\begin{aligned} &\sum_{i,j=1}^{N} (a_{ij}^1(x) u_{0x_j} + a_{ij}^2(x) v_{0x_j}) \cos(n, x_i) \,|_{\partial\Omega - \Gamma_1} = 0 \\ &\sum_{i,j=1}^{N} (a_{ij}^3(x) u_{0x_j} + a_{ij}^4(x) v_{0x_j}) \cos(n, x_i) \,|_{\partial\Omega - \Gamma_2} = 0 \end{aligned}\right\} \tag{3.7}$$

thus

$$\mathcal{A}_{11}u_0(\varphi)+\mathcal{A}_{12}v_0(\varphi)=\int_\Omega-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}(a^1_{ij}(x)u_{0x_j}+a^2_{ij}(x)v_{0x_j})\bar{\varphi}dx$$

$$+\int_{\partial\Omega}\sum_{i,j=1}^N(a^1_{ij}u_{0x_j}+a^2_{ij}v_{0x_j})\bar{\varphi}\cos(n,x_i)ds$$

That is, $\mathcal{A}_{11}u_0(\psi)+\mathcal{A}_{12}v_0(\psi)$ is a linear continuous functional on $L^2(\Omega)$. In the same way, we see $\mathcal{A}_{21}u_0(\psi)+\mathcal{A}_{22}v_0(\psi)$ is a linear continuous functional on $L^2(\Omega)$ thus $\mathcal{A}[u_0,v_0]\in W$.

Summarize the above and then Theorem 1 assert the existence and uniqueness of a solution of the problem:

$$C_{11}(x)\frac{\partial^2}{\partial t^2}u(x,t)+C_{12}(x)\frac{\partial^2}{\partial t^2}v(x,t)+R(x)\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial\mu}\frac{\partial}{\partial t}u(x,t)$$

$$+\sum_{j=1}^N b_j(x)\frac{\partial}{\partial t}v_{x_j}(x,t)+b_0(x)\frac{\partial}{\partial t}v(x,t)-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}\Big(a^1_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)$$

$$+a^2_{ij}(x)\frac{\partial}{\partial x_j}v(x,t)\Big)=F_1(x,t) \tag{3.8}$$

$$C_{21}(x)\frac{\partial^2}{\partial t^2}u(x,t)+C_{22}(x)\frac{\partial^2}{\partial t^2}v(x,t)+\sum_{j=1}^N\bar{b}_j(x)u_{x_j}(x,t)-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}\Big(a^3_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)$$

$$+a^4_{ij}(x)\frac{\partial}{\partial x_j}v(x,t)\Big)=F_2(x,t),\quad (x,t)\in\Omega\times(0,T),\ a.e. \tag{3.9}$$

$$\left.\begin{aligned}&\sum_{i,j=1}^N(a^1_{ij}(x)u_{x_j}+a^2_{ij}(x)v_{x_j})\cos(n,x_i)\,|_{\partial\Omega-\Gamma_1}=0\\ &\qquad\qquad\qquad\qquad\qquad\qquad (t\geqslant 0)\\ &\sum_{i,j=1}^N(a^3_{ij}(x)u_{x_j}+a^4_{ij}(x)v_{x_j})\cos(n,x_i)\,|_{\partial\Omega-\Gamma_2}=0\end{aligned}\right\} \tag{3.10}$$

$$u(x,t)=0,\ x\in\Gamma_1,\ v(x,t)=0,\ x\in\Gamma_2,\ t\geqslant 0 \tag{3.11}$$

$$u(x,0)=u_0(x),\ v(x,0)=v_0(x), \tag{3.12}$$

$$\frac{\partial u}{\partial t}(x,0)=u_1(x),\ \frac{\partial v}{\partial t}(x,0)=v_1(x) \tag{3.13}$$

(3.8),(3.9) and (3.10) are true under the meaning of variations. But u_{tt} and $v_{tt}\in L^2(\Omega)$, $u_t\in V_1$, $v_t\in V_2$. (3.10) is true in $L^2(\partial\Omega-\Gamma_1)$ and $L^2(\partial\Omega-\Gamma_2)$ respectively, if the regularity of such boundary value problem is guaranteed.

IV. Cauchy Problem for (1.4)

Suppose:

$$\left.\begin{aligned}&|a(t,u,v)|\leqslant k_1\|u\|_V\|v\|_V,\ (k_1>0,\ u,\ v\in V)\\ &\operatorname{Re}a(t,u,v)\geqslant c_1\|u\|_V^2,\ (u\in V,\ c_1>0)\end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned}&|b(t,u,v)|\leqslant k_2\|u\|_V\|v\|_V,\ (k_2>0,\ u,\ v\in V)\\ &\operatorname{Re}b(t,u,u)\geqslant c_2\|u\|_V^2,\ (c_2>0,\ u\in V)\end{aligned}\right\} \tag{4.2}$$

Define $V_m = V\times W$, the product Hilbert space with scalar-product given by

$$([x_1,\ x_2],\ [y_1,\ y_2])_{V_m}=(x_1,\ y_1)_V+(x_2,\ y_2)_W \qquad ([x_1,\ x_2],\ [y_1,\ y_2]\in V_m)$$

Let C and $\mathscr{C}$ represent the Riesz maps from V and W to V' and W' respectively. Define $M\in \mathscr{L}(V_m,\ V_m')$ by

$$\mu[x_1,\ x_2]([y_1,\ y_2])=([x_1,\ x_2],\ [y_1,\ y_2])_{V_m},\ [x_1,\ x_2],\ [y_1,\ y_2]\in V_m$$

so M is the Riesz map from V_m to V_m',

Let $v=u'$, from

$$\begin{bmatrix} C & \\ & \mathscr{C}\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}'+\begin{bmatrix} & -C \\ \mathscr{A}(t) & B(t)\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}=\begin{bmatrix} 0 \\ f(t)\end{bmatrix} \quad (t\geqslant 0) \tag{4.3}$$

we can obtain (1.4). And, using an elementary transformation, (4.3) is equivalent to

$$A(t)=M^{-1}L(t),\ g(t)=M^{-1}F(t),$$
$$D(A(t))=D(L(t))\triangleq\{[x_1,x_2]\in V\times V:\mathscr{A}(t)x_1+B(t)x_2\in W'\}$$

then equivalent to (4.4),

$$\begin{bmatrix} C & \\ & \mathscr{C}\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}'+\begin{bmatrix} \lambda C & -C \\ \mathscr{A}(t) & B(t)+\lambda\mathscr{C}\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}=\begin{bmatrix} 0 \\ f(t)e^{-\lambda t}\end{bmatrix} \quad (t\geqslant 0) \tag{4.4}$$

Let $W(t)=[u(t),\ v(t)]$, denote the operator in the second term of left-hand side of (4.4) by $L(t)$ and the right-hand side by $F(t)$. Let

$$W'(t)+A(t)W(t)=g(t) \qquad (t\geqslant 0) \tag{4.5}$$

where $D(A(t))=\{[x_1,x_2]\in V\times V:\mathscr{A}(t)x_1+B(t)x_2\in W'\}$.

This is a standard form of first order linear evolution equation.

Assume $D(A(t))$ is a fixed subspace, so that some theorems in [2] are applicable to prove the existence and uniqueness of a solution of Cauchy's problem for (4.5).

Definition Let H be a Hilbert space. If $D(A)$ is dense in H, A is accretive, and $\lambda+A$ is surjective for some $\lambda>0$, then A is called maximum accretive operator.

Lemma 4 Let $A\in L(D(A),\ H)$ be a maximum accretive operator, a Hilbert space V dense in H, $V\hookrightarrow H$ continuously. Assume $a(u,v)$ is continuous sesquilinear V-elliptic form, moreover,

$$a(u,v)=(Au,v)_H,\ u\in D(A),\ \forall v\in V$$

then $-A$ is the generator of an analytic semigroup driven by $a(.,.)$.

Proof For every $f\in H'$, there exists only one solution $u\in V$ such that $a(u,v)=f(v)$, $(\forall v\in V)$.

Then we have a linear operator T and a subspace $D(T)$ with

$$a(u,v)=(Tu,v)_H \qquad (u\in D(T),\quad v\in V) \tag{4.6}$$

According to [1], $-T$ is the generator of an analytic semigroup

(4.6) shows T is an extension of A but A is maximum and $A=T$.

Define $E([\ \ ],\ [\ \ ])$ by

$$E([u,v],\ [\varphi,\ \psi])=\lambda(u,\varphi)_V-(v,\varphi)_V+a(t,u,\psi)+b(t,v,\psi)+\lambda(v,\psi)_W$$
$$([u,v],\ [\varphi,\psi]\in V\times V) \tag{4.7}$$

It is clear that $V\times V$ is dense and continuously imbeded in V_m.

Lemma 5 $E([\ \ ].[\ \ ])$ is a continuous sesquilinear elliptic form on $V\times V$, with

$$(A(t)[u,v],\ [\varphi,\psi])_{V_m}=E([u,v],\ [\varphi,\psi])$$
$$([u,v]\in D(A(t)),\quad [\varphi,\psi]\in V\times V) \tag{4.8}$$

Proof From (4.7), we have

$$|E([u,v],\ [\varphi,\psi])|\leqslant M\|[u,v]\|_{V\times V}\|[\varphi,\psi]\|_{V\times V}, \tag{4.8$'$}$$

For $|\mathrm{Re}(a(t,u,v)-(v,u)_V)|\leqslant\beta\|u\|\cdot\|v\|\quad(\beta>0)$.

We have

$$\mathrm{Re}E([u,v],\ [u,v])$$
$$\geqslant\lambda\|u\|^2+c_2\|v\|^2-\beta\|u\|\,\|v\|\geqslant\lambda\|u\|^2+c_2\|v\|^2-\frac{\beta\eta}{2}\|u\|^2-\frac{\beta}{2\eta}\|v\|^2 \tag{4.9}$$

If we choose large η such that $c_2>\beta/2\eta$ and choose λ such that $\lambda>\beta\eta/2$,we can obtain elliptic property of E by (4.9). That is

$$|E([u,v],\ [u,v])|\geqslant\alpha\|[u,v]\|^2_{V\times V} \tag{4.10}$$

Because

$$(A(t)[u,v],[\varphi,\psi])_{Vm}=(\mu^{-1}L(t)[u,v],\ [\varphi,\psi])_{Vm}=L(t)[u,v]([\varphi,\psi])$$
$$=\lambda(u,\varphi)_V-(v,\varphi)_V+\mathscr{A}(t)u(\psi)+B(t)v(\psi)+\lambda(v,\psi)_W$$
$$([u,v]\in D(A(t)),\quad [\varphi,\psi]\in V_m) \tag{4.11}$$

we know (4.8) is true.

Lemma 6 $A(t)$ is maximum accretive.

Proof In order to prove that for every $[f_1,f_2]\in V_m$, there exists an $[x_1,\ x_2]\in D(A(t))$ such that $A(t)[x_1,x_2]=[f_1,f_2]$ we need only to prove that there exists $[x_1, x_2]$ such that

$$\left.\begin{array}{l}\lambda cx_1-cx_2=cf_1,\\ \mathscr{A}(t)x_1+B(t)x_2+\lambda\mathscr{C}x_2=\mathscr{C}f_2\end{array}\right\} \tag{4.12}$$

We can find $x_2\in V$ such that

$$\left(\frac{1}{\lambda}\mathscr{A}(t)+B(t)+\lambda\mathscr{C}\right)x_2=\mathscr{C}f_2-\frac{1}{\lambda}\mathscr{A}(t)f_1$$

and let

$$x_1=x_2/\lambda+f_1/\lambda\in V$$

We can see $[x_1, x_2]$ satisfies (4.11) and $\in D(A(t))$. Thus $A(t)$ is a surjection of $D(A(t))$onto V_m.

From (4.10) and (4.9), we have

$$\|A(t)[u,v]\|\geqslant\alpha\|[u,v]\|,\quad ([u,v]\in D(A(t)))\ ,\ \alpha>0.$$

Thus $A^{-1}(t)$ exists and $\|A^{-1}(t)\|\leqslant 1/\alpha$

Take η such that $0<\eta<\alpha$. Let

$$P[u,v]=-\eta A^{-1}(t)[u,v]+A^{-1}(t)[f_1,f_2],$$

Then P is a constraction. And P has a fixed point $[u,v]$, such that $(A(t)+\eta)[u,v]=[f_1,f_2]$ Thus $(\eta+A(t))$ is surjective onto V_m.

It is easy to show that $A(t)$ is accretive.

We denote by $S(\theta)$ a fixed closed sector which consists of those complex numbers η satisfying $-\theta\leqslant arg\ \lambda\leqslant\theta,\ \ \theta>\pi/2$.

Lemma 7 The resolvent set $\rho(A(t))$ contains $S(\theta)$ and the resolvent $(\eta+A(t))^{-1}$ satisfies

$$\|(\eta I+A(t))^{-1}\|\leqslant\frac{c}{1+|\eta|},\quad \eta\in S(\theta),\quad \|\eta\|\geqslant 1 \tag{4.13}$$

Proof Because $E([u,v],\ [\varphi,\psi])$ is a continuous sesquilinear elliptic form and M in (4.8)$'$ and α in (4.10) are independent of η and t, from (4.8), we have, by [1],

$$\|\eta(\eta+A(t))^{-1}\|\leqslant M(\theta)\qquad(\eta\in S(\theta)) \tag{4.14}$$

where $M(\theta)$ is independent of t and η.

Thus we have (4.13) by (4.14) and C is independent of t and η

Lemma 8 Denote

$$L_0=\begin{bmatrix}\lambda c & -c\\ \mathscr{A}(t) & B(t)\end{bmatrix},\quad L=\begin{bmatrix}\lambda c & -c\\ \mathscr{A}(t) & B(t)+\lambda\mathscr{C}\end{bmatrix},$$

Then both L_0 and L are maximum accretive operators of $D(A(t))$ with

$$\|L_0[u,v]\|_{V'_m}\geqslant\alpha\|[u,v]\|_{V\times V},\quad \|L[u,v]\|_{V'_m}\geqslant\alpha\|[u,v]\|_{V\times V}$$
$$[u,v]\in D(A(t)) \tag{4.15}$$

The proof is similar to that of Lemma 6.

Lemma 9 Assume operator $B^{-1}(t)\cdot\mathscr{A}(t)$ and subspace $B^{-1}(t)W'\subset V$ are independent of t. If $B(s)\cdot B^{-1}(t)$ is a continuous surjection from V' onto itself, and its restriction on w' satisfies

$$\|B(s)B^{-1}(t)-B(r)B^{-1}(t)\|_{\mathscr{L}(W')}\leqslant c|s-r| \tag{4.16}$$

then we have

$$\|A(s)A^{-1}(t)-A(r)A^{-1}(t)\|_{\mathscr{L}(V_m)}\leqslant\tilde{c}|s-r|\qquad(\tilde{c}>0) \tag{4.17}$$

Proof We have

$$\begin{aligned}\|(A(s)-A(r))[x_1,x_2]\|_{V_m}&=\|(L(s)-L(r))[x_1,x_2]\|_{V'_m}\\&=\|\mathscr{A}(s)x_1+B(s)x_2-\mathscr{A}(r)x_1-B(r)x_2\|_{W'}\\&=\|B(s)(x_2+B^{-1}(s)\mathscr{A}(s)x_1)-B(r)(x_2+B^{-1}(r)\mathscr{A}(r)x_1)\|_{W'}\end{aligned} \tag{4.18}$$

Denote $y=x_2+B^{-1}(t)\mathscr{A}(t)x_1$, and then y is independent of t.

From (4.16) we have

$$\|B(s)y-B(r)y\|_{W'}\leqslant c\|B(t)y\|_W|s-r|$$

i.e.

$$\|\mathscr{A}(s)x_1+B(s)x_2-\mathscr{A}(r)x_1-B(r)x_2\|_{W'}\leqslant c\|\mathscr{A}(t)x_1+B(t)x_2\|_{W'}\ |s-r| \quad (4.19)$$

By Lemma 8, we can define a norm on $D(A(t))$ by $\|[x_1, x_2]\|_{D(A(t))}=\|L[x_1,x_2]\|_{V'_m}$, $[x_1,x_2]\in D(A(t))$, which makes $D(A(t))$ a Banach space. Since L is a closed operator and (4.15). And so does L_0. Then it follows that for some positive numbers C_1 and C_2

$$c_1\|L_0[\quad]\|_{V'_m}\leqslant\|L[\quad]\|_{V'_m}\leqslant c_2\|L_0[\quad]\|_{V'_m} \quad (4.20)$$

Then it follows that

$$\|\mathscr{A}(t)x_1+B(t)x_2\|_{W'}\leqslant\|L_0[x_1,x_2]\|_{V'_m}\leqslant\frac{1}{c_1}\|L[x_1,x_2]\|_{V'_m},\ [x_1,x_2]\in D(A(t)) \quad (4.21)$$

From (4.21), (4.19) we obtain (4.17).

We summarize the above.

Theorem 2 Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume $\mathscr{A}(t)$ and $B(t)\in\mathscr{L}(V,V')$ defined by (4.1), (4.2) and (1.5). If $B(t)$ satisfies (4.16), $B^{-1}(t)\mathscr{A}(t)$ and $B^{-1}(t)W'$ are independent of t, then for every $f\in C^1(R_0^+,W')$ and $[u_0,u_0]\in V\times V$ with $\mathscr{A}(0)u_0+B(0)u_1\in W'$, there exists a unique solution $u(t)$ of

$$\left.\begin{array}{l}\mathscr{C}u''(t)+B(t)u'(t)+\mathscr{A}(t)u(t)=f(t)\\ u(0)=u_0,\ u'(0)=u_1\end{array}\quad (t\geqslant 0)\right\} \quad (4.22)$$

Proof $A(t)$ is maximum accretive by Lemma 6 and $-A(t)$ is the generator of an analytic semigroup by Lemma 4 and 5. From Lemma 7 we have (4.13). We obtain (4.17) by Lemma 9. So finally Theorem 2 is proved.

Corollary 1 In addition to the hypotheses of theorem 2, assume that $B(t)=\varepsilon\mathscr{A}(t)$, $\varepsilon>0$. Then for every $f\in C^1(R_0^+,W')$ and $[u_0,u_1]\in V\times V$ with $\mathscr{A}(t)(u_0+\varepsilon u_1)\in W'$, there exists a unique solution $u(t)$ of

$$\left.\begin{array}{l}\mathscr{C}u''(t)+\varepsilon\mathscr{A}(t)u'(t)+\mathscr{A}(t)u(t)=f(t)\\ u(0)=u_0,\ u'(0)=u_1\end{array}\quad t\geqslant 0\right\} \quad (4.23)$$

It is clear that $B^{-1}(t)\mathscr{A}(t)$ is independent of t.

Corollary 2 Under the conditions of theorem 1, if $\mathscr{A}(t)=0$, then for every $f\in C(R_0^+,W')$, $u_0\in V$, $u_1\in B^{-1}(t)W'$, there exists a unique solution $u(t)$ of

$$\left.\begin{array}{l}\mathscr{C}u''(t)+B(t)u'(t)=f(t)\\ u(0)=u_0,\ u'(0)=u_1\end{array}\quad t\geqslant 0\right\} \quad (4.24)$$

It is clear that $B^{-1}(t)\mathscr{A}(t)$ is independent of t.

V. Discussion About the Boundary Problem (4.23)

It is natural to ask a question: May ε in (4.23) go to 0?

If for every s, $A(t)A^{-1}(s)$ is twice differentiable in t and $f'(\cdot)$ is a Hoelder continuous function, then, by [2], solution of (4.5) and therefore solution of (4.23) is differentiable in high order.

So from (4.23), we have

$$\begin{aligned}&(u'''(t),u''(t))_W+\varepsilon a(t,u''(t),u''(t))+a(t,u'(t),u''(t))\\&+\varepsilon a'(t,u'(t),u''(t))+a'(t,u(t),u''(t))=f'(t)(u''(t))\end{aligned} \quad (5.1)$$

For

$$2\mathrm{Re}\,(u'''(t),u''(t))_W=\frac{d}{dt}\|u''(t)\|_W^2,\quad 2\mathrm{Re}a(t,u''(t),u''(t))\geqslant 0$$

$$2\mathrm{Re}a\,(t,u'(t),u''(t))=\frac{d}{dt}a(t,u'(t),u'(t))-a'(t,u'(t),u'(t))$$

$$2\mathrm{Re}a'(t,u'(t),u''(t))=\frac{d}{dt}a'(t,u'(t),u'(t))-a''(t,u'(t),u'(t))$$

$$2\mathrm{Re}a'(t,u(t),u''(t))=2\,\mathrm{Re}\frac{d}{dt}a'(t,u(t),u'(t))$$

$$-2\mathrm{Re}a''(t,u(t),u'(t))-2\mathrm{Re}a'(t,u'(t),u'(t))$$

so

$$\Big[\frac{d}{dt}\|u''(t)\|_W^2+2\varepsilon\mathrm{Re}a(t,u''(t),u''(t))+\frac{d}{dt}a(t,u'(t),u'(t))$$
$$-a'(t,u'(t),u'(t))+\varepsilon\frac{d}{dt}a'(t,u'(t),u'(t))-\varepsilon a''(t,u'(t),u'(t))$$
$$+2\,\mathrm{Re}\frac{d}{dt}a'(t,u(t),u'(t))-2\mathrm{Re}a''(t,u(t),u'(t))$$
$$-2\mathrm{Re}a'(t,u'(t),u'(t))\Big]=2\mathrm{Re}f'(t)(u''(t)) \tag{5.2}$$

Integrate (5.2) from 0 to t, we have

$$[\|u''(t)\|_W^2-\|u''(0)\|_W^2+a(t,u'(t),u'(t))-a(0,u'(0),u'(0))$$
$$+\varepsilon a'(t,u'(t),u'(t))-\varepsilon a'(0,u'(0),u'(0))$$
$$+2\mathrm{Re}a'(t,u(t),u'(t))-2\mathrm{Re}a'(0,u(0),u'(0))]$$
$$\leqslant 2\,\mathrm{Re}\int_0^t\|f'(\tau)\|_{W'}\|u''(\tau)\|_W d\tau+3\int_0^t a'(\tau,u'(\tau),u'(\tau))d\tau$$
$$+\varepsilon\int_0^t a''(\tau,u'(\tau),u'(\tau))d\tau+2\mathrm{Re}\int_0^t a''(\tau,u(\tau),u'(\tau))d\tau \tag{5.3}$$

so

$$\|u''(t)\|_W^2+a(t,u'(t),u'(t))\leqslant\|u''(0)\|_W^2+a(0,u_1,u_1)$$
$$+\|f'\|^2_{L^2(0,T;W')}+\int_0^t(\|u''(\tau)\|_W^2+c\|u'(\tau)\|_V^2)d\tau \tag{5.4}$$

where $c=3\|\mathscr{A}'(t)(\mathscr{L}V,V')$

By choosing $\mathscr{A}(0)u_0\in W'$, $\mathscr{A}(0)u_1\in W'$, from (4.23) we have

$$\|u''(0)\|_W\leqslant\|f(0)\|_{W'}+\varepsilon\|\mathscr{A}(0)u_0\|_{W'}+\|\mathscr{A}(0)u_0\|_{W'} \tag{5.5}$$

Therefore

$$\|u''(t)\|_W^2+c_1\|u'(t)\|_V^2\leqslant\beta(\|f(0)\|_{W'}^2+\varepsilon^2\|\mathscr{A}(0)u_0\|_{W'}^2$$
$$+\|\mathscr{A}(0)\,u_0\|_{W'}^2+\|u_1\|_V^2+\|f'\|^2_{L^2(0,T;W')})$$
$$+\int_0^t(\|u''(\tau)\|_W^2+c\|u'(\tau)\|_V^2)d\tau\qquad(\beta>0) \tag{5.6}$$

When $\varepsilon \to 0$, right first term in (5.6) is bounded. So by Gronwall's lemma we have

$$\|u''(t)\|_W^2 + c\|u'(t)\|_V^2 \leqslant M + c(T)M \tag{5.7}$$

where $C(T)$ is only dependent on T.

$u''(t)$ and $u'(t)$ come from (4.23), so are dependent on ε. M is independent of ε. Thus (5.7) shows passing limit as $\varepsilon \to 0$ in (4.23) is meaningful.

Summarize the above, we have

Theorem 3 Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume that $\mathscr{A}(t) \in \mathscr{L}(V, V')$ satisfies elliptic and symmetric conditions and (4.16). If $\|\mathscr{A}(t)\|_{\mathscr{L}(V,V')}$, $\|\mathscr{A}'(t)\|_{\mathscr{L}(V,V')}$, and $\|\mathscr{A}''(t)\|_{\mathscr{L}(V,V')}$ are all bounded independent of t, for every $x \in \mathscr{A}^{-1}(s)W'$, $\mathscr{A}'(t)x$, $\mathscr{A}''(t)x \in C^1(R_0^+, W')$. Then for every $f(\cdot) \in C^1(R_0^+, W')$, $u_0 \in V$, $u_1 \in V$, with $\mathscr{A}(0)u_0 \in W'$ and $\mathscr{A}(0)u_1 \in W'$, there exists a unique solution of

$$\begin{cases} \mathscr{C}u''(t) + \mathscr{A}(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \qquad (t \geqslant 0)$$

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