

## ON SOME PROBLEMS FOR UNSYMMETRICAL LATERAL BUCKLING OF RECTANGULAR PLATES

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### Abstract

*The present paper investigates several problems for unsymmetrically lateral instability of rectangular plates by the energy method. In the text we discuss the minimum critical load of rectangular plates which possess the unsymmetrical supporters and to which the lateral buckling occurs unsymmetrically under a concentrated force, uniformly distributed load and the concentrated couples respectively.*

### I. Introduction

Since to write the differential equations by which the lateral buckling of plates is represented will involve great mathematical difficulties, it would be more suitable to study the problems for lateral instability of plates by means of the energy method. For the purpose of solving the minimum critical load while the rectangular plates have the unsymmetrical lateral buckling, we use the Rayleigh-Ritz's method to find out the least load when the general potential energy of the buckling system arrives at a stationary value. We may divide the process of lateral buckling of thin plate into two stages. Let the coordinates be taken as shown in Fig. 1.

In the first stage the thin plate is bent in the vertical plane; in the second stage the thin plate produces the torsion and deflects from the primary vertical plane and is accompanied with the lateral bending. Evidently the thin plate, which is bent in the vertical plane in the first stage, maintains its equilibrium state with external load in the flat form; because the rigidity of the plate in the middle plane (in the vertical plane) is greater, the general potential energy, which corresponds to the bending of thin plate within the vertical plane in the first stage, may be considered approximately as zero. Then this portion may be neglected in general potential energy, when we analyse the whole process of the lateral buckling of thin plate. Therefore we need only investigate the general potential energy of thin plate in the second stage. This is the same as in the stability problems, in which the straight bar is compressed, and the axial deformations of the bar before bending are not considered.

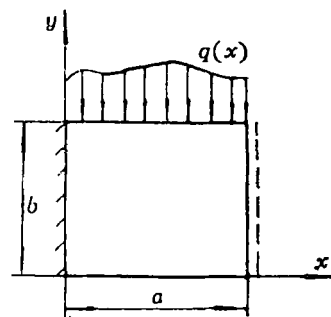


Fig. 1

### II. Method

Letting an isotropic rectangular plate with constant thickness, the strain potential energy for the total system in bending is<sup>[1]</sup>

$$U = \frac{D}{2} \iint [w_{xx}^2 + w_{yy}^2 + 2\mu w_{xx}w_{yy} + 2(1-\mu)w_{xy}^2] dx dy \quad (2.1)$$

in which  $D$  is the flexural rigidity of the plate and  $D = Eh^3/12(1-\mu^2)$ ,  $E$ ,  $h$ ,  $\mu$  are the modulus of elasticity of the materials of the plates, the thickness of the plates, the Poisson's ratio respectively.  $w$  is a flexural function;  $w_{xx}$ ,  $w_{yy}$ , and so forth are the partial derivatives of the flexural function with respect to the corresponding lower subscriptions. The above double integrations are all over the region of the medium surface of the plate.

If the thin plate (see Fig. 1) buckles under the distributed loading in medium plane on a top side ( $y = b$ ), then the potential energy of external forces may be represented as

$$V = - \int_0^a q(x) \Delta_y(x) dx \quad (2.2)$$

where  $q(x)$  is the distributed loading along the  $x$ -axis acting along the  $y$ -direction in the middle plane of the plate.  $\Delta_y(x)$  are the displacements at the acting points of the loading  $q$  in vertical directions (along the  $y$ -axis) when the plate has the lateral bending. They are the functions of the coordinate  $x$ . The above integration is performed along the boundary ( $y = b$ ), on which the loading is applied.

The general potential energy of system is then

$$\Pi = U + V \quad (2.3)$$

In light of Rayleigh-Ritz's method<sup>[1]</sup>, when the system arrives at the limit in stable equilibrium the general potential energy is a minimum, i.e.

$$\delta \Pi = 0 \quad (2.4)$$

Consequently we may determine the minimum critical loading.

The flexural function in formula (2.1) should be selected in advance to satisfy all the geometrical boundary conditions of thin plates.

### III. Several Cases for Unsymmetrical Lateral Buckling of Rectangular Plates

**Case A** Let a rectangular plate be clamped on the one edge and be simply supported on the opposite edge. The other pair of edges in the opposite direction are free, subjected by a concentrated force  $P$  vertically at the mid-point  $E(a/2, b)$  on the top edge (free edge  $y = b$ ) as shown in Fig. 2.

Now we choose the flexural functions as follows

$$w = \left( f_1 + \frac{f_2 - f_1}{b} y \right) \left( \cos \frac{3\pi x}{2a} - \cos \frac{\pi x}{2a} \right) \quad (3.1)$$

in which  $a$  and  $b$  are the lengths of the edges of thin plate along the  $x$ - and the  $y$ -axes respectively,  $f_1$  and  $f_2$  are the maximum lateral deflections on the lower and the upper free edges of rectangular plate respectively. They are the unknown parameters. It can be easily seen that function (3.1) can satisfy the whole geometrical boundary conditions of this thin plate.

Firstly we calculate the strain potential energy of thin plate. For this end, substituting function (3.1) into (2.1) and performing integration we may obtain

$$U = \frac{1}{2} \left[ (A+B)f_1^2 + (A+B)f_2^2 + (A-2B)f_1 f_2 \right] \quad (3.2)$$

in which

$$A = \frac{41}{48} \pi^4 \frac{b}{a^3} D, \quad B = \frac{5}{2} \pi^2 \frac{1-\mu}{ab} D \quad (3.3)$$

Secondly we calculate the potential energy of external force  $V$ , since  $\Delta_y(x)$  in formula (2.2) may be known from geometrical analysis

$$\Delta_y(x) = \frac{1}{2} (w \cdot w_y)_E \quad (3.4)$$

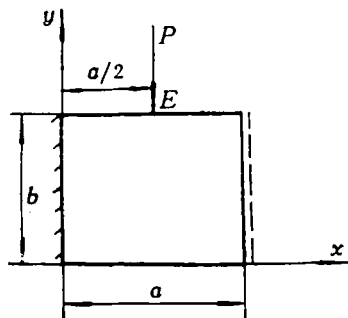


Fig. 2

in which the lower subscription  $m$  denotes that the quantity in the bracket must take the value at the point  $E(a/2, b)$ .

The potential energy of external forces may be calculated by formula (2.2), if we observe (3.4) then we obtain

$$V = -\frac{1}{2} P (w \cdot w_y)_E$$

Substituting (3.1) into the above formula we may obtain

$$V = -\frac{1}{b} P (f_1^2 - f_1 f_2) \quad (3.5)$$

Substituting (3.2) and (3.5) into (2.3) and from (2.4) performing variation with respect to the two parameters  $f_1$  and  $f_2$  we may obtain the following homogeneous linear systems of the algebraic equations related to the parameters  $f_1$  and  $f_2$

$$\left. \begin{aligned} 2(A+B)f_1 + (A-2B+2C)f_2 &= 0 \\ (A-2B+2C)f_1 + 2(A+B-2C)f_2 &= 0 \end{aligned} \right\} \quad (3.6)$$

in which

$$C = P/b \quad (3.7)$$

To find the nontrivial solutions of systems (3.6) the determinant of the coefficients in the above equations must be equal to zero. Then we obtain the equation of the stability of the plate

$$4C^2 + 12AC - 3A(A+4B) = 0 \quad (3.8)$$

from which we may find out its minimum positive real root

$$C = \frac{1}{2} (-3A + 2\sqrt{3A(A+B)}) \quad (3.9)$$

Observe (3.3) and (3.7) we may obtain the minimum critical loading of lateral buckling for this rectangular plate under a concentrated force  $P$

$$P_{cr} = \frac{\pi^2 D}{4a} \left\{ -\frac{41}{8} \pi^2 \frac{b^2}{a^2} + \pi \frac{b}{a} \sqrt{41 \left[ \frac{41}{48} \pi^2 \frac{b^2}{a^2} + \frac{5}{2} (1-\mu) \right]} \right\} \quad (3.10)$$

For the case of square plate, then  $b/a = 1$ , if we take  $\mu = 0.3$ , we may obtain

$$P_{cr} = 33.56036 D/a \quad (3.11)$$

This numeral is just between  $18.44293 D/a$  and  $41.80256 D/a$ , which is the minimum critical loading

for the square plates simply supported and clamped on each pair of supported edges respectively.

**Case B** Let a rectangular plate whose support is the same as in Case A and be subjected by a uniformly distributed load of intensity  $q_0$  on the top side (free side  $y=b$ ) as indicated in Fig. 3.

In this question the flexural function still applies formula (3.1), hence the strain potential energy is also the same as in formula (3.2). Using formula (2.2) we may calculate the potential energy of external forces. Taking note of (3.4), substituting (3.1) into (2.2) and performing integration we may obtain

$$V = -\frac{1}{2} \frac{a}{b} q_0 (f_1 - 2f_2) \quad (3.12)$$

Proceeding with operation as in Case A we may obtain the minimum critical loading for the rectangular plate, in which the lateral buckling under a uniformly distributed load of intensity  $q_0$  occurs.

$$(q_0)_{cr} = \frac{\pi^2}{2a^2} D \left\{ -\frac{41}{8} \pi^2 \frac{b^2}{a^2} + \pi \frac{b}{a} \sqrt{41 \left[ \frac{41}{48} \pi^2 \frac{b^2}{a^2} + \frac{5}{2} (1-\mu) \right]} \right\} \quad (3.13)$$

For the case of square plate,  $b/a = 1$ , if we take  $\mu = 0.3$ , then we get

$$(q_0)_{cr} = 67.12072 D/a^2 \quad (3.14)$$

such a numeral is just between  $36.88483 D/a^2$  and  $111.47349 D/a^2$ , which is the minimum critical loading for the square plates simply supported at all and clamped on each pair of supported edges respectively.

In the above two cases, if we lower the level to be acted on by the load  $P$  or  $q_0$ , for instance, from  $y=b$  to  $y=b/2$  or to  $y=0$ , then the minimum critical loading having been produced the unsymmetrical lateral buckling of thin plate will increase.

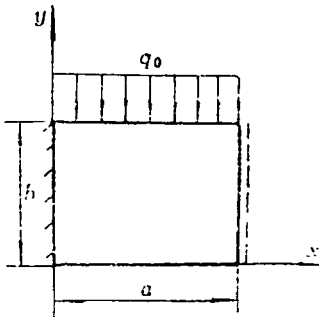


Fig. 3

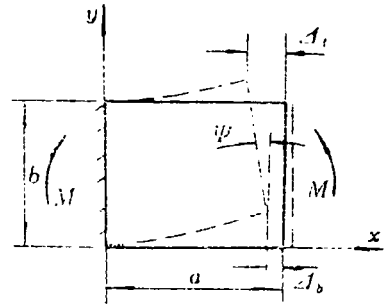


Fig. 4

**Case C** Let a rectangular plate whose support is the same as in the above two cases and be subjected by a couple  $M$  on each end to be supported as denoted in Fig. 4.

In this case the flexural functions and the strain potential energy are still the same as in formulae (3.1) and (3.2) respectively. The potential energy of external forces should be calculated by the following formula

$$V = -M\Psi \quad (3.15)$$

The angle  $\Psi$  is shown in the figure and

$$\psi = (\Delta_t - \Delta_b)/b \quad (3.16)$$

in which the meanings of  $\Delta_t$  and  $\Delta_b$  are also indicated in Fig. 4 and its value is calculated by using the following formulae

$$\Delta_t = \frac{1}{2} \int_0^a (w_x^2)_{x=0} dx, \quad \Delta_b = \frac{1}{2} \int_0^a (w_x^2)_{x=a} dx \quad (3.17)$$

Substituting the flexural function (3.1) into (3.17) and proceeding with integration, by (3.16) and (3.15) we may obtain

$$V = -\frac{5}{8} \frac{\pi^2}{ab} M (f_2^2 - f_1^2) \quad (3.18)$$

Proceeding with calculation as the above we may obtain the minimum critical loading of the lateral buckling for certain rectangular plate under a pair of the concentrated couple  $M$

$$M_{cr} = \frac{\pi}{10} \frac{b}{a} D \sqrt{41 \left[ \frac{41}{48} \pi^2 \frac{b^2}{a^2} + 10(1-\mu) \right]} \quad (3.19)$$

For the square plate,  $b/a = 1$ , by taking  $\mu = 0.3$  we may obtain

$$M_{cr} = 7.90184D \quad (3.20)$$

This numeral is also just between  $4.68347D$  and  $13.60691D$ , which is the minimum critical loading for the square plates simply supported and clamped on each pair of supported edges respectively.

#### IV. Concluding Remarks

The present paper gives the method of calculation of the minimum critical loading of unsymmetrical lateral buckling in several loaded cases for rectangular plates consisting of the unsymmetrical supporters. Every value of the critical loading found depends on the flexural rigidity of the plates and the ratio of the length of the sides. Since the flexural functions chosen are not sole, the answer to be obtained by different flexural functions will not be the same generally. Whether the approximate solutions are good or bad chiefly depends on the flexural functions to be selected by us.

#### References

- [1] Timoshenko, S., *Theory of Elastic Stability* (1936).
- [2] Cheng Xiang-sheng, Lateral instability of rectangular plates, *Applied Math. and Mech.*, **10**, 1 (1989), 87-92.