

ON THE OBJECTIVE STRESS RATE IN CO-MOVING COORDINATE SYSTEM

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Abstract

The objective stress rate is a rather important problem in mechanics of finite deformation. In this paper, the objective stress rate in co-moving coordinate is derived by applying nonlinear geometric field theory of deformation. Problems, such as large extension coupled with rotation, and large shear deformation, are exemplified by using the new formula. Comparing with Jaumann's stress rate and other formulae presented in current literature, the new result appears to be the reasonable one in co-moving coordinate system.

I. Introduction

For large displacement field, objective stress rate appears to play an important role in constitutive law in mechanics of finite deformation. Due to different choices of the reference frames for finite deformation, various expressions of the definition of objective stress rate have been derived in recent years, such as Jaumann's, Green-Naghdi's, Truesdell's. The main purpose of these objective rates lies in keeping the true stress rate for large rotation and large strain in the process of deformation. Recently, Atluri^[2] discussed this problem and suggested a proposal in solving the arguments.

In the present paper, a new formula of objective (or true) stress rate in co-moving coordinate system is derived by using the nonlinear geometrical field theory of Chen^[10]. The applicability of the new definition is verified by practical examples.

II. Short Account on the Current Definitions of Objective Stress Rate

Let \mathbf{F} be the deformation gradient, \mathbf{L} the velocity gradient. We decompose \mathbf{L} into two parts: \mathbf{D} , the symmetric component of \mathbf{L} is deformation rate tensor, and \mathbf{W} , the antisymmetric component of \mathbf{L} is rotation rate tensor,

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D} + \mathbf{W} \quad (2.1)$$

By the theorem of polar decomposition, another variety of rotation rate tensor may be defined as

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T, \quad \mathbf{F} = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U} \quad (2.2)$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress referred to a deformed configuration. Then the Jaumann (J), Green-Naghdi (GN) and Truesdell (T) objective rates of stress are defined respectively as:

$$\dot{\mathbf{v}}^J = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W} \quad (2.3)$$

$$\dot{\mathbf{v}}^{GN} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega} \quad (2.4)$$

$$\dot{\mathbf{v}}^T = \dot{\boldsymbol{\sigma}} - \mathbf{L}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{L}^T + \boldsymbol{\sigma}\text{tr}(\mathbf{D}) \quad (2.5)$$

Although the expressions for (J), (GN), (T) objective stress rates are different, there are relations between them; they are defined based on Cauchy stress $\boldsymbol{\sigma}$, unrotated stress $\boldsymbol{\tau}$, and Piola-Kirchhoff stress \mathbf{S} respectively, the relations are:

$$\boldsymbol{\sigma} = (1/J)\mathbf{F}\mathbf{S}\mathbf{F}^T \quad (2.6)$$

$$\boldsymbol{\tau} = (1/J)\mathbf{U}\mathbf{S}\mathbf{U} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R} \quad (2.7)$$

with

$$J = \det(\mathbf{F}) > 0$$

Three definitions are derived based on the same assumption, i.e., the transformation from one configuration to another is obtained by adding a rigid rotation. Let \mathbf{X} , \mathbf{X}^* represent the position vectors in two adjacent configurations, then

$$\mathbf{X}^* = \mathbf{a}(t) + \mathbf{R}(t)\mathbf{X} \quad (2.8)$$

where $\mathbf{a}(t)$ is rigid translation and $\mathbf{R}(t)$ is orthogonal tensor representing a rigid rotation. It is quite evident that to identify the local rotation of a deforming body as a rigid rotation is not justifiable, since it is unreasonable to express real deformation rotation by the concept of rigid rotation in a deforming process.

Studies and discussions have been made to examine which definition is more reasonable. Bazant^[3] pointed out that Jaumann stress rate is objective, but no finite strain and finite strain rate conjugate with it. On the other hand, Johnson and Banmann^[6] emphasized that Jaumann stress rate is unacceptable for its instability. Recently, Molenskump discussed specially the "Limit to the Jaumann stress rate"^[7], and shows that Jaumann stress rate is not accurate for moderate deformation with deviative strain more than 10%; in such cases the objective stress rate is better calculated using the material rotation rate as by polar decomposition theorem. Through an analysis of problem of large simple shear, Moss^[8] concluded that the solutions for (J), (GN), (T) stress rates are not monotonous and unstable. The instability of solution for physical problems lies in the unreasonable mathematical structure for real physical model.

We shall abandon the assumption of adding a rigid rotation in derivation of objective stress rate. In the present paper, we derive true stress rate directly from the definition of stress in co-moving coordinate system.

III. Derivation of Objective (True) Stress Rate in Co-Moving System

Consider an elemental surface area ABC in a Local System, da_i is covariant components of the area, \mathbf{g}^i is basic vectors

$$\begin{aligned} d\mathbf{A} &= da_i \mathbf{g}^i \\ da_i &= \frac{1}{4} \epsilon_{ijk} dx^j dx^k \end{aligned} \quad (3.1)$$

where ϵ_{ijk} is permutation tensor, x^i are co-moving coordinate.

Let \mathbf{T} be traction vector acting on a surface element, σ^i stress vectors on coordinate surfaces, and σ^{ij} , σ^i_j the components of stress tensors,

$$\mathbf{T} = \mathbf{t} dA = \sigma^i da_i = \sigma^{ij} da_j \mathbf{g}_j = \sigma^i_j da_j \mathbf{g}^j \quad (3.2)$$

Consider first the time derivative of σ^i_j

$$\dot{\mathbf{T}} = \dot{\sigma}^i_j da_j \mathbf{g}^j + \sigma^i_j da_j \dot{\mathbf{g}}^j + \sigma^i_j d\dot{a}_j \mathbf{g}^j \quad (3.3)$$

where $\dot{}$ indicates $\partial/\partial t$

Since

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \quad (3.4)$$

it follows

$$\dot{\mathbf{g}}^i \cdot \mathbf{g}_j = -\mathbf{g}^i \cdot \dot{\mathbf{g}}_j$$

It is easy to verify that

$$\dot{\mathbf{g}}_j = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}}{\partial x^j} \right) = \frac{\partial \mathbf{V}}{\partial x^j} = V^i \|_{,j} \mathbf{g}_i \quad (3.5)$$

where $()_{,j}$ denotes covariant derivative with respect to x^j in current configuration.

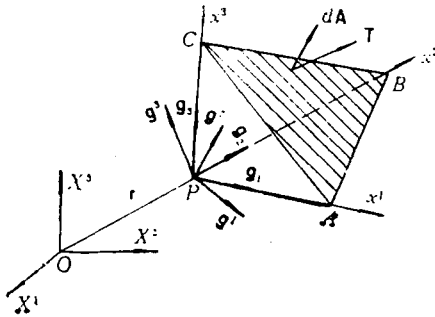


Fig. 1

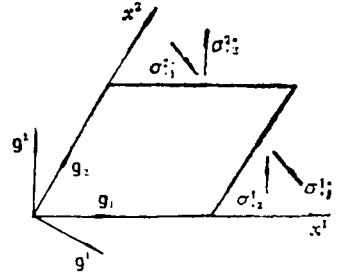
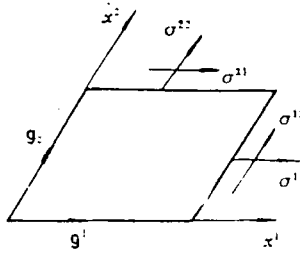


Fig. 2

Combining the above results, we have

$$\dot{\mathbf{g}}^j = -V^j \|_{,i} \mathbf{g}^i \quad (3.6)$$

Moreover,

$$d\dot{a}_i = \frac{1}{4} \frac{\partial}{\partial t} (\epsilon_{ijk}) dx^j dx^k = \frac{1}{4} \delta_{ijk} \frac{\partial}{\partial t} (\sqrt{g}) dx^j dx^k \quad (3.7)$$

As

$$\frac{\partial}{\partial t} \sqrt{g} = \frac{\partial}{\partial t} [\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)] = V^i \|_{,i} \sqrt{g} \quad (3.8)$$

We get

$$d\dot{a}_i = V^i \|_{,i} da_i \quad (3.9)$$

Finally, Eq. (3.3) becomes

$$\dot{\mathbf{T}} = (\dot{\sigma}_{ij} + \sigma_{ij} V^k_{|i} - \sigma_{ij} V^k_{|j}) da_i g^j \quad (3.10)$$

It is evident from the geometrical change of local frame in the time interval, the time rate of stress is dependent on local strain. It is proved in [1], by applying S-R decomposition theorem, the velocity gradient in current configuration can be decomposed as

$$V^k_{|j} = L^k_{|j} \dot{\vartheta} + \dot{S}^k_{|j}, \quad V^k_{|i} = \dot{S}^k_{|i} \quad (3.11)$$

where $\dot{S}^k_{|j}$ and $L^k_{|j} \dot{\vartheta}$ are rate of finite strain and finite rotation referred to current configuration respectively. Then Eq. (3.10) becomes:

$$\dot{\mathbf{T}} = [\dot{\sigma}_{ij} - \sigma_{ij} \dot{S}^k_{|i} + \sigma_{ij} \dot{S}^k_{|j}] da_i g^j - \sigma_{ij} L^k_{|j} \dot{\vartheta} da_i g^j \quad (3.12)$$

Next, we shall show here that the last term in the above equation expresses the effect of mean solid rotation of a deforming body as a whole. It is induced by the change of orientation of \mathbf{T} , and should be eliminated since it is not the true stress rate.

When a body element $PABC$ (Fig. 3) rotates as a whole, the rate of change of \mathbf{T} due to mean solid rotation is to be

$$\dot{\mathbf{T}}_s = \mathbf{L} \dot{\vartheta} \times \mathbf{T} = \mathbf{L} \dot{\vartheta} \times \sigma_{ij} da_i g^j \quad (3.13)$$

a simple calculation shows

$$\dot{\mathbf{T}}_s = \sigma_{ij} L^k_{|j} \dot{\vartheta} da_i g^j = -\sigma_{ij} L^k_{|i} \dot{\vartheta} da_i g^j \quad (3.14)$$

so the true stress rate is found to be

$$\nabla^k_{|j} \equiv \dot{\sigma}_{ij} - \sigma_{ij} \dot{S}^k_{|i} + \sigma_{ij} \dot{S}^k_{|j} \quad (3.15)$$

$$\dot{\mathbf{T}}_t = \dot{\mathbf{T}} - \dot{\mathbf{T}}_s = \nabla^k_{|j} da_i g^j \quad (3.16)$$

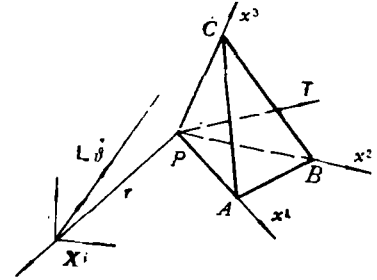


Fig. 3

In the same way, we have

$$\nabla^{ij} \equiv \dot{\sigma}^{ij} + \sigma^{ij} \dot{S}^k_{|i} + \sigma^{ij} \dot{S}^k_{|j} \quad (3.17)$$

As we know tensor components are generally not with common physical dimension, gauge transformation is necessary for the constitutive components in the formulae of true stress rate turning into physical component for practical computation (Some discussions on new definition of stress physical component are seen in [9]).

In physical components:

$$\hat{\nabla}^k_{|j} = \hat{\sigma}^k_{|j} - \hat{\sigma}^k_{|i} \hat{S}^i_{|j} + \hat{\sigma}^k_{|j} \hat{S}^i_{|i} \quad (3.18)$$

$$\hat{\nabla}^{ij} = \hat{\sigma}^{ij} + \hat{\sigma}^{ij} \hat{S}^k_{|i} + \hat{\sigma}^{ij} \hat{S}^k_{|j} \quad (3.19)$$

It can be seen from the formulae that true stress rate includes three parts: the first is the change of magnitude itself, the next is the effect of change of elemental configuration, and the third is the effect of change of acting area.

It should be emphasized in some literature that the objective (true) stress rate is expressed in the form of Eq. (3.10); however, we have shown here that this is not true at least in co-moving frame.

For incremental computation, if we neglect the effect of geometric change in every step, common (or material) rate of stress may be used,

$$\left. \begin{aligned} \Delta \hat{\sigma}^k_{|j} &= \int_t^{t+\Delta t} \hat{\sigma}^k_{|j} dt \\ \Delta \hat{\sigma}^{ij} &= \int_t^{t+\Delta t} \hat{\sigma}^{ij} dt \end{aligned} \right\} \quad (3.20)$$

For nonlinear elasticity, the incremental constitutive equation is to be in the following form

$$\left. \begin{aligned} \hat{\nabla} &= \mathbf{C} : \hat{\mathbf{S}} \\ \hat{\nabla}_i &= C_{ij} \hat{S}_j, \hat{\nabla}^{ij} = C^{ijkl} \hat{S}_{kl} \end{aligned} \right\} \quad (3.21)$$

In general, the material coefficients may not be constants, but in many cases, constant coefficients will be useful for good approximation.

IV. Examples of Verification

a. Unilateral large extension

The transformation function is

$$X^1 = (1 + \phi)x^1, \quad X^2 = x^2, \quad X^3 = x^3$$

where X^i is the coordinates of a material point referred to fixed space Cartesian frame, x^i is co-moving coordinates, $t=0$, $x^i = X^i$.

Deformation gradient

$$[F] = \begin{bmatrix} 1+\phi & 0 \\ 0 & 1 \end{bmatrix}, \quad [F]^{-1} = \begin{bmatrix} 1/(1+\phi) & 0 \\ 0 & 1 \end{bmatrix}$$

Metric tensor

$$[g_{ij}] = \begin{bmatrix} (1+\phi)^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Velocity gradient

$$[V_{ij}] = [\dot{F}] = \begin{bmatrix} \dot{\phi} & 0 \\ 0 & 0 \end{bmatrix}, \quad [V^i_{||j}] = [F]^{-1}[\dot{F}] = \begin{bmatrix} \frac{\dot{\phi}}{1+\phi} & 0 \\ 0 & 0 \end{bmatrix}$$

Physical components of velocity gradient

$$[\hat{V}^i_{||j}] = \begin{bmatrix} \frac{\dot{\phi}}{1+\phi} & 0 \\ 0 & 0 \end{bmatrix} = [\hat{S}^i_{||j}]$$

The rate type elastic law may be cast into the form:

$$\hat{\nabla}_i = 2G \left(\delta_i^j \delta_j^k + \frac{\nu}{1-2\nu} \delta_i^k \delta_j^j \right) \hat{S}_k$$

where G —elastic shear modulus, ν —Poisson's ratio.

By equation (3.18), we have

$$\hat{\nabla}_1 = \frac{\partial \hat{S}_1}{\partial t} = (2G + \lambda) \dot{\phi} / (1 + \phi) \quad \left(\lambda \equiv 2G \frac{\nu}{1-2\nu} \right)$$

$$\hat{\nabla}_2 = \frac{\partial \hat{S}_2}{\partial t} + \hat{S}_2^j \dot{\phi} / (1 + \phi) = \lambda \dot{\phi} / (1 + \phi)$$

$$\hat{\nabla}_3 = \frac{\partial \hat{S}_3}{\partial t} + \hat{S}_3^j \dot{\phi} / (1 + \phi) = \lambda \dot{\phi} / (1 + \phi)$$

$$\hat{\nabla}_i = 0 \quad i \neq j$$

with initial condition

$$\hat{\sigma}_i^j|_{t=0}=0 \quad (i, j=1, 2, 3)$$

The solution of this set of differential equations gives

$$\hat{\sigma}_1^1 = (2G + \lambda) \ln(1 + \phi), \quad \hat{\sigma}_2^2 = \hat{\sigma}_3^3 = \lambda \frac{\phi}{1 + \phi}, \quad \hat{\sigma}_i^j = 0 \quad i \neq j$$

This example elucidates the effect of the change of acting area during large extension process. During the time interval, from $t=t^0$ to t' , the area of acting surface of $\hat{\sigma}_1^1$ does not change, however, the areas of acting surface of $\hat{\sigma}_2^2$ and $\hat{\sigma}_3^3$ enlarge, i.e., $A_2' = (1 + \phi)A_2^0$, $A_3' = (1 + \phi)A_3^0$, $A_1' = A_1^0$.

b. Large extension coupled with large rotation

The transformation function is

$$\begin{aligned} X^1 &= [1 + \phi(t)] \cos \theta(t) x^1 - \sin \theta(t) x^2 \\ X^2 &= [1 + \phi(t)] \sin \theta(t) x^1 + \cos \theta(t) x^2 \\ X^3 &= x^3 \end{aligned}$$

where $\phi = \phi(t)$ is the elongation in direction of co-moving coordinate x^1 , and $\theta = \theta(t)$ is rotation angle.

Deformation gradient

$$\begin{aligned} [F] &= \begin{bmatrix} (1 + \phi) \cos \theta & -\sin \theta \\ (1 + \phi) \sin \theta & \cos \theta \end{bmatrix} \\ [F]^{-1} &= \begin{bmatrix} \frac{\cos \theta}{1 + \phi} & \frac{\sin \theta}{1 + \phi} \\ -\sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

Metric tensor

$$[g_{ij}] = \begin{bmatrix} (1 + \phi)^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Velocity gradient

$$\begin{aligned} [V^i_{,j}] &= \begin{bmatrix} -\dot{\theta}(1 + \phi) \sin \theta + \dot{\phi} \cos \theta, & -\dot{\theta} \cos \theta \\ \dot{\theta}(1 + \phi) \cos \theta + \dot{\phi} \sin \theta, & -\dot{\theta} \sin \theta \end{bmatrix} \\ [V^i_{||j}] &= \begin{bmatrix} \frac{\dot{\phi}}{1 + \phi} & -\frac{\dot{\theta}}{1 + \phi} \\ (1 + \phi)\dot{\theta} & 0 \end{bmatrix} \end{aligned}$$

Physical component of velocity gradient

$$[\hat{V}^i_{||j}] = \begin{bmatrix} \frac{\dot{\phi}}{1 + \phi} & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}$$

Physical component of strain rate

$$[\hat{S}^i_{||j}] = \begin{bmatrix} \frac{\dot{\phi}}{1 + \phi} & 0 \\ 0 & 0 \end{bmatrix}$$

We see here that the same result has been arrived at as the case of pure extension in example (a).

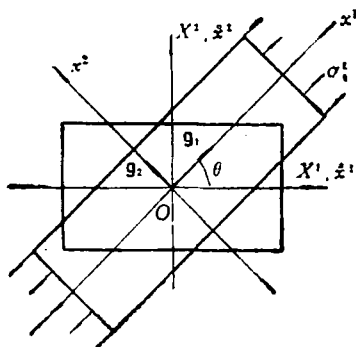


Fig. 4

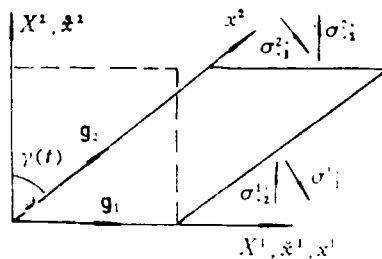


Fig. 5

Mean solid rotation has no influence on the stress solution in co-moving system.

We confirm again that only Eqs. (3.18) or (3.19) can be used to obtain correct solution.

c. Large simple shear deformation

Problem of large simple shear is generally used to test the correctness of various definitions of objective stress rate in current literature.

The transformation function is

$$X^1 = x^1 + k(t)x^2, \quad X^2 = x^2, \quad X^3 = x^3$$

where

$$k(t) = \tan \gamma(t)$$

Deformation gradient

$$[F] = \begin{bmatrix} 1 & k(t) \\ 0 & 1 \end{bmatrix}, \quad [F]^{-1} = \begin{bmatrix} 1 & -k(t) \\ 0 & 1 \end{bmatrix}$$

Metric tensor

$$[g_{ij}] = \begin{bmatrix} 1 & k \\ k & 1 + k^2 \end{bmatrix}$$

Velocity gradient

$$[V^i_{;j}] = \begin{bmatrix} 0 & \dot{k} \\ 0 & 0 \end{bmatrix},$$

$$[V^i_{||j}] = [F]^{-1} [V^i_{;j}] = \begin{bmatrix} 0 & \dot{k} \\ 0 & 0 \end{bmatrix}$$

Physical component of velocity gradient

$$[\hat{V}^i_{||j}] = \begin{bmatrix} 0 & \dot{k} / \sqrt{1 + k^2} \\ 0 & 0 \end{bmatrix}$$

Physical component of strain rate

$$[\hat{S}^i_{;j}] = \begin{bmatrix} 0 & \frac{1}{2} \frac{\dot{k}}{\sqrt{1 + k^2}} \\ \frac{1}{2} \frac{\dot{k}}{\sqrt{1 + k^2}} & 0 \end{bmatrix}$$

By Eq. (3.18) and incremental elastic constitutive law, we get

$$\frac{\partial \hat{\sigma}_1^1}{\partial t} = \frac{1}{2} \hat{\sigma}_1^1 \dot{k} / \sqrt{1+k^2}, \quad \frac{\partial \hat{\sigma}_1^2}{\partial t} = \frac{1}{2} \hat{\sigma}_1^2 \dot{k} / \sqrt{1+k^2}$$

$$\frac{\partial \hat{\sigma}_1^1}{\partial t} = G \dot{k} / \sqrt{1+k^2} + \frac{1}{2} \hat{\sigma}_1^1 \dot{k} / \sqrt{1+k^2}$$

$$\frac{\partial \hat{\sigma}_1^2}{\partial t} = G \dot{k} / \sqrt{1+k^2} + \frac{1}{2} \hat{\sigma}_1^2 \dot{k} / \sqrt{1+k^2}$$

With initial condition

$$\hat{\sigma}_i^j|_{t=0} = 0$$

Putting $d\tilde{k} = dk / \sqrt{1+k^2}$, the solution of the above set of differential equations gives

$$\hat{\sigma}_1^1 = \hat{\sigma}_1^2 = G(\exp[\tilde{k}/2] - \exp[-\tilde{k}/2])$$

$$\hat{\sigma}_1^1 = \hat{\sigma}_1^2 = G(\exp[\tilde{k}/2] + \exp[-\tilde{k}/2]) - 2G$$

The final results are

$$\hat{\sigma}_1^1 = \hat{\sigma}_1^2 = G[(k + \sqrt{1+k^2})^{\frac{1}{2}} + (k - \sqrt{1+k^2})^{-\frac{1}{2}} - 2]$$

$$\hat{\sigma}_1^1 = \hat{\sigma}_1^2 = G[(k + \sqrt{1+k^2})^{\frac{1}{2}} - (k - \sqrt{1+k^2})^{-\frac{1}{2}}]$$

The values of $\hat{\sigma}_1^1$, $\hat{\sigma}_1^2$ are shown in Table 1. and Fig. 6.

The second order effect of cross elasticity appears in the solution. When $\gamma(t)$ is small enough, $\hat{\sigma}_1^1$, $\hat{\sigma}_1^2$ are high order infinitesimal quantities, and can be neglected. But with $\gamma(t)$ increasing, $\hat{\sigma}_1^1$, $\hat{\sigma}_1^2$ gradually become in the same order as $\hat{\sigma}_1^1$, $\hat{\sigma}_1^2$, and may have predominant effect infracture phenomena. It is seen that the new definition of true stress rate derives a solution that is monotonous and stable.

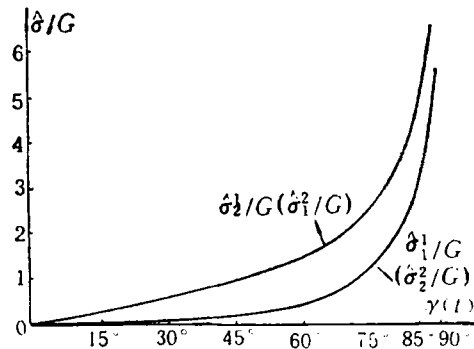


Fig. 6

Table 1 The values of $\hat{\sigma} - \gamma(t)$

$\gamma(t)$	15°	30°	45°	60°	75°	85°
$\hat{\sigma}_1^1/G$	0.017	0.076	0.197	0.449	1.119	2.995
$\hat{\sigma}_1^2/G$	0.266	0.566	0.910	1.414	2.393	4.577

V. Comparison of Results

Many different forms of objective (true) stress rate have been derived, but none of them can

Table 2

No.	Classification	Formula of true stress rate	Solution of large elastic shear problem	Note
1	Present paper (Chen-Shang)	$\dot{\mathbf{v}} = \frac{D\boldsymbol{\sigma}}{Dt} - \boldsymbol{\sigma} \cdot \dot{\mathbf{S}} + \boldsymbol{\sigma} \text{tr}(\mathbf{e})$ (in co-moving coordinate)	$\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = G \left[(k + \sqrt{1+k^2})^{1/2} + (k + \sqrt{1+k^2})^{-1/2} - 2 \right]$ $\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = G \left[(k + \sqrt{1+k^2})^{1/2} - (k + \sqrt{1+k^2})^{-1/2} \right]$ $k \equiv \text{tg} \gamma(t)$ (Physical components)	$\mathbf{T} = \sigma_1^i \dot{\gamma}^i d\mathbf{a}_i \mathbf{g}^i$ $= \sigma_1^i d\mathbf{a}_i \mathbf{g}^i$ $\hat{\mathbf{S}}^i = \frac{1}{2} (\dot{\gamma}^i \cdot \mathbf{1}_i + \dot{\gamma}^i \cdot \mathbf{1}_i^T)$
2	Jaumann	$\dot{\mathbf{v}} = \frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau} \cdot \mathbf{W} - \mathbf{W} \cdot \boldsymbol{\tau}$	$\tau_{11} = -\tau_{22} = G[1 - \cos k(t)]$ $\tau_{12} = \tau_{21} = G \sin k(t)$	$\mathbf{e} \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{L} + \mathbf{W}$
3	Truesdell	$\dot{\mathbf{v}}_1 = \frac{D\boldsymbol{\tau}}{Dt} - \mathbf{L} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}' + \boldsymbol{\tau} \text{tr}(\mathbf{e})$	$\tau_{11} = 4Gk^2, \quad \tau_{22} = 0$ $\tau_{12} = \tau_{21} = Gk$	$\dot{\mathbf{v}}_1 = \frac{D\dot{\mathbf{v}}_1^{\text{vN}}}{Dt} - \mathbf{g}^M \mathbf{g}_N$
4	Green-Naghdi	$\dot{\mathbf{v}} = \frac{D\boldsymbol{\tau}}{Dt} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega}$	$\tau_{11} = -\tau_{22} = 4G(\cos 2\beta \ln(\cos \beta) + \beta \sin 2\beta - \sin^2 \beta)$ $\tau_{12} = \tau_{21} = 2G \cos 2\beta (2\beta - 2\text{tg} 2\beta \ln(\cos \beta) - \text{tg} \beta)$ $\beta = \text{tg}^{-1}(k/2)$	$\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^T = -\dot{\mathbf{R}} \cdot \mathbf{R}'$ $\mathbf{F} = \mathbf{R}\mathbf{U}$
5	Cotter-Rivlin	$\dot{\mathbf{v}}_2 = \frac{D\boldsymbol{\tau}}{Dt} + \mathbf{e}' \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{e}$	$\tau_{11} = 0, \quad \tau_{22} = -Gk^2$ $\tau_{12} = \tau_{21} = Gk$	$\dot{\mathbf{v}}_2 = \frac{D\dot{\mathbf{v}}_2^{\text{vN}}}{Dt} - \mathbf{g}^M \mathbf{g}_N$
6	Mixed-hybrid I	$\dot{\mathbf{v}}_3 = \frac{D\boldsymbol{\tau}}{Dt} - \mathbf{e}' \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{e}$	$\tau_{11} = G(1 - \cos k), \quad \tau_{22} = 0$ $\tau_{12} = G \sin k, \quad \tau_{21} = Gk$	$\dot{\mathbf{v}}_3 = \frac{D\dot{\mathbf{v}}_3^{\text{M}^*}}{Dt} - \mathbf{g}^M \mathbf{g}_N$
7	Mixed-hybrid II	$\dot{\mathbf{v}}_4 = \frac{D\boldsymbol{\tau}}{Dt} + \mathbf{e}' \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{e}$	$\tau_{11} = G(1 - \cos k), \quad \tau_{22} = 0$ $\tau_{12} = Gk, \quad \tau_{21} = G \sin k$	$\dot{\mathbf{v}}_4 = \frac{D\dot{\mathbf{v}}_4^{\text{M}^*}}{Dt} - \mathbf{g}^M \mathbf{g}_N$

describe the true stress rate properly. In order to get the correct result, two main points must be kept in mind:

(1) A rational choice of description method for the motion of a deforming body. In the present paper, stress is defined in a co-moving (embedding) system, so we can describe the change intrinsically.

(2) The concept of rigid rotation is not applicable to a body while it is deforming. A discussion of nonclassical concept of rotation for deforming body is seen in [11].

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