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SET-VALUED CARISTI'S FIXED POINT THEOREM AND EKELAND'S VARIATIONAL PRINCIPLE

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Abstract

This paper proposes a formally stronger set-valued Caristi's fixed point theorem and by using a simple method we give a direct proof for the equivalence between Ekeland's variational principle and this set-valued Caristi's fixed point theorem. The results stated in this paper improve and strengthen the corresponding results in [4].

In 1974, Ekeland[1] proposed the following variational principle:

Theorem 1 (Ekeland's variational principle[1]. Let (X,d) be a complete metric space and $\varphi:X\longrightarrow(-\infty,+\infty]$ a lower semi-continuous functional bounded from below and $\not\equiv+\infty$. Suppose that ε is an arbitrary position number and u a point in X such that

$$\varphi(u)\leqslant\inf\{\varphi(x):\ x\in X\}+\varepsilon \tag{1}$$

Then for any $\lambda>0$ there exists a point v in X such that the following hold

$$\varphi(v)\leqslant\varphi(u)-\varepsilon\lambda\cdot d(u,\ v) \tag{2}$$

$$d(u,\ v)\leqslant\frac{1}{\lambda} \tag{3}$$

$$\varphi(x)>\varphi(v)-\varepsilon\lambda\cdot d(v,\ x)\qquad\forall\, x\in X\setminus\{v\} \tag{4}$$

In 1976, Caristi[3] proposed the following fixed point theorem:

Theorem 2 (single-valued Caristi's fixed point theorem[3] Let (X,d) be a complete metric space and $\varphi:\ X\longrightarrow[0,\infty)$ a lower semi-continuous function. Suppose that $T:\ X\to X$ is a mapping satisfying the following condition:

$$d(x,\ Tx)\leqslant\varphi(x)-\varphi(Tx),\qquad\forall\, x\in X$$

Then T has a fixed point in X.

It is well known that these two famous theorems are of fundamental importance in the recent theory of nonlinear analysis. In particular, they play an important role in the control theory, optimization, global analysis, geometric theory of Banach spaces and nonlinear semigroups etc.

By virtue of Theorem 1, in 1979, Ekeland[2] gave a direct proof of Theorem 2. Recently, using Theorem 2, in 1987, Shi[4] obtained a direct proof of Theorem 1. Hence the problem of how to give a direct proof for the equivalence between Ekeland's variational principle and Caristi's fixed point theorem is solved.

The purpose of this paper is to obtain the following formally stronger set-valued Caristi's fixed point theorem 3 and by using a simple method we give a direct proof for the equivalence between Ekeland's variational principle and this set-valued Caristi's fixed point theorem. The results stated in this paper improve and strengthen the corresponding results in [4].

Theorem 3 Let (X,d) be a complete metric space, $CB(X)$ a family of all nonempty subsets of X and $\varphi: X\to(-\infty, +\infty]$ a lower semi-continuous functional bounded from below and $\not\equiv +\infty$. Suppose that $T: X \to CB(X)$ is a mapping satisfying the following condition: for any $x\in X$ there exists a $y\in Tx$ such that

$$d(x, y)\leqslant\varphi(x)-\varphi(y) \tag{5}$$

Then for any $u\in X$, $\varphi(u)\neq+\infty$, and $\beta>1$, there exists a fixed point $v\in X$ of T such that

$$d(u, v)\leqslant\beta(\varphi(u)-\varphi(v)) \tag{6}$$

In particular, if

$$\varphi(u)\leqslant\inf\{\varphi(x): x\in X\}+\varepsilon<\inf\{\varphi(x):x\in X\}+1$$

then v has the following property:

$$d(u, v)\leqslant\sqrt{\varepsilon} \tag{7}$$

Now we give the main result of this paper:

Theorem 4 Theorem 1 and Theorem 3 are equivalent to each other.

Proof Theorem 1 $\Longrightarrow$ Theorem 3.

For any $u\in X$, $\varphi(u)\neq+\infty$ and $\beta>1$, if

$$\varphi(u)=\inf\{\varphi(x): x\in X\}$$

then $\varphi(u)\leqslant\varphi(y)$, $\forall y\in Tu$. By condition (5) we get, for some $y\in Tu$,

$$d(u, y)=0$$

This implies that $u\in Tu$, and u is a fixed point of T. Taking $v=u$, hence we know that (6), and (7) are true.

If

$$\varphi(u)>\inf\{\varphi(x): x\in X\}$$

denote

$$\varphi(u)-\inf\{\varphi(x): x\in X\}=\varepsilon$$

Using Theorem 1 for φ, and taking $\lambda=(\varepsilon\beta)^{-1}$, it follows that there exists a $v\in X$ such that

$$\varphi(v)\leqslant\varphi(u)-\frac{1}{\beta}d(u, v) \tag{8}$$

$$d(u, v)\leqslant\varepsilon\beta \tag{9}$$

$$\varphi(x)\geqslant\varphi(v)-\frac{1}{\beta}d(v, x), \qquad \forall x\in X \tag{10}$$

By condition (5) it follows from (10) that there exists $y\in Tv$ such that

$$d(v, y)+\varphi(y)\leqslant\varphi(v)\leqslant\frac{1}{\beta}d(v, y)+\varphi(y)$$

Since $\beta>1$ and $\varphi(v)\leqslant\varphi(u)<+\infty$, it follows from the above inequality that

$$d(v, y)=0$$

This shows that $v=y\in Tv$, i.e. v is a fixed point of T. In addition, it follows from (8) that (6) is true.

Besides, if $0<\varepsilon<1$, taking $\beta=(\varepsilon)^{-1/2}>1$ and noting (9) it gets (7).

This completes the proof.

Theorem 3 $\Longrightarrow$ Theorem 1.

Proof by contradiction. Suppose that under the conditions of Theorem 1 the assertions of Theorem 1 are wrong, hence there exists a $\lambda>0$ such that for any $x\in X$

$$\varphi(x)>\varphi(u)-\varepsilon\lambda d(u, x) \tag{11}$$

$$d(u, x)>\frac{1}{\lambda} \tag{12}$$

and a y_0, $y_0\neq x$ such that

$$\varphi(y_0)\leqslant\varphi(x)-\varepsilon\lambda d(x, y_0) \tag{13}$$

Letting

$$F(x)=\{y\in X:\ y\neq x \text{ and } \varphi(y)+\varepsilon\lambda d(x, y)\leqslant\varphi(x)\},\qquad x\in X$$

it follows from (13) that $y_0\in F(x)$, hence $F(x)$ is nonempty. Next, by the lower semi-continuity of φ we know that $F(x)$ is a subset of X. This implies that F is a mapping from X into $CB(X)$ and it has no fixed point in X.

On the other hand, by the definition of F, for any $x\in X$ we have

$$d(x, y)\leqslant\frac{1}{\varepsilon\lambda}(\varphi(x)-\varphi(y)),\qquad \forall y\in F(x) \tag{14}$$

By Theorem 3 F has a fixed point in X. This is a contradiction. By this contradiction we pove that (13) is wrong. Besides, it is obvious that (11) and (12) can not hold, either. This means that under the conditions of Theorem 1, for any $\lambda>0$ there exists a $v\in X$ such that the assertions (2) and (3), (4) hold.

This completes the proof of Theorem 4.

References

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