

ON THE STABILITY OF DISTORTED LAMINAR FLOW (I) —BASIC IDEAS AND THEORY*

Zhou Zhe-wei (周哲玮)

(Shanghai Institute of Applied Mathematics and Mechanics, Shanghai)

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Abstract

This paper suggests a hydrodynamic stability theory of distorted laminar flow, and presents a kind of distortion profile of mean velocity in parallel shear flow. With such distortion profiles, the new theory can be used to investigate the stability behaviour of parallel shear flow, and thus suggests a new possible approach to instability.

1. Introduction

In 1883, Reynolds showed through his well known Reynolds experiments that the fluid has different types of motion—laminar and turbulent^[1]. The physical or mathematical explanation of this phenomenon, however, makes the most prominent scientists fall into a difficult position. Reynolds then said^[2]: “The theory itself appears to be tolerably complete—and affords the means of calculating the results to be expected in almost every case of fluid motion, but while in many cases the theoretical results agree with those actually obtained, in other cases they are altogether different.”

By a hundred years' exploration, the scientists have not yet achieved a substantial improvement on the situation when the problem was first proposed. Now we have some excellent experiments, showing important features of some cases of transition process or turbulence; and have several theoretical concepts, some of which have been checked by experiments.

The observation of turbulence suggests two problems of different nature: The understanding of the cause that leads the smoothly flowing fluid into turbulence; and the understanding of the behaviour of the fully developed turbulent flow. This paper only involves the first problem.

The instability of laminar flow is the first stage of the generation of turbulence, this idea was suggested by Reynolds in 1883, but he attributed it to Stokes (1843). Tollmien once said^[3]: “It is certain that this fundamental change in type of motion of the fluid is traceable to an instability in the laminar flow, for laminar flows of themselves would always be possible solutions of the hydrodynamic equations.” This paper investigates the stability behaviour of parallel shear flow, the explanation of the physical mechanism of the instability in such a flow is very difficult to ascertain. In an exact sense, parallel shear flow only includes pipe flow (Hagen-Poiseuille flow, Fig. 1) and the flow between parallel planes (plane Couette flow, Fig. 2; plane Poiseuille flow, Fig. 3). Boundary layer flow can also be regarded as a parallel shear flow under the local parallel assumption, but this flow is not considered in this paper.

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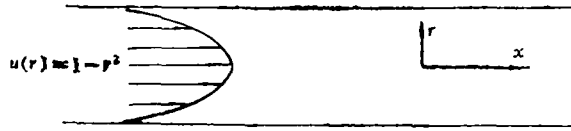


Fig. 1 Hagen – Poiseuille flow



Fig. 2 Plane Couette flow

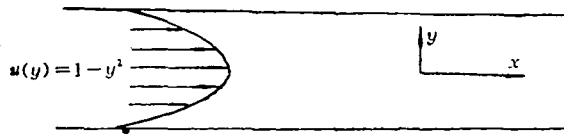


Fig. 3 Plane Poiseuille flow

Through his experiments in a circular pipe, Reynolds discovered the existence of two types of motions of a viscous fluid — laminar and turbulent, and suggested a dimensionless parameter $Re = ua/\nu$ as a criterion indicating the transition from laminar to turbulent flow, where u is the velocity at the center of the pipe, a is the radius of the pipe, and ν is the kinematic viscosity coefficient. This dimensionless parameter is called Reynolds number, which is a characteristic number denoting kinematic similarity of fluid motion. We can define the Reynolds number of different flow by choosing relevant characteristic velocity and length. Some people select mean velocity and the diameter of the pipe as the characteristic velocity and length, in this paper we adopt the maximum velocity and the radius of the pipe. For plane Poiseuille flow, the characteristic length is half the distance between two parallel boundaries, and the characteristic velocity is the velocity at the center; for plane Couette flow, half the distance between two parallel boundaries is chosen as the characteristic length, half the relative shifting velocity of two boundaries as the characteristic velocity.

The critical Reynolds numbers indicating the transition in the series of experiments made by Reynolds is about 13000^[4], the critical values range from 2000 to 10^5 in later pipe flow experiments. Leite (1959)^[5] is the first who investigates the development of artificial controlled disturbance in a circular pipe. In his experiment, the flow can sustain laminar motion until $Re \approx 13000$. In this range, Leite found that all the infinitesimal disturbances damp out, and along the axis the decay rate of the nonaxisymmetric components is faster than that of the axisymmetric one. So small nonaxisymmetric disturbances should be more stable than axisymmetric one. In the same experiment, if we disturb the laminar flow stronger (mounting in the pipe a circular ring with apex angle which can shift periodically along the axis), when $Re = 4000$, turbulence will occur if the oscillating amplitude of the circular ring exceeds some critical value (Kuethe (1956)^[6]). Fox, Lessen, Bhat (1968)^[7] observed unstable nonaxisymmetric disturbance with azimuthal wave number equal to one. In their experimental apparatus, the laminar motion can be kept till $Re = 5000$, from the critical Reynolds number 2130 they obtained, the disturbances have possibly finite amplitude.

In the laboratory, Davis and White (1928)^[8], Kao and Park (1970)^[9], and Patel (1969)^[10] have shown the plane Poiseuille flow is unstable to finite-amplitude disturbances at Reynolds numbers as low as 1000. On the other hand, Nishoka, Iida and Ichikawa (1975)^[11] performed experiments in a low-turbulence wind-tunnel in which they were able to maintain laminar plane Poiseuille flow at Reynolds number as large as 8000. Nishoka, et al. had to reduce the background turbulence level to less than 0.05%. At larger disturbance levels, instability is obtained at lower (subcritical) Reynolds numbers. They also observed the threshold amplitude of finite-amplitude disturbances, and thus gave evidence of nonlinear hydrodynamic stability theory.

The experimental situation with regard to plane Couette flow is far less satisfactory. Robertson (1959)^[12] and Reichardt (1959)^[13] showed that a turbulent flow is obtained at sufficiently large Reynolds number. Although the accurate value of Reynolds numbers when the laminar flow becomes unstable can not be obtained, it is between 600 and 1450 in Reichardt's experiment.

It appears that the transition Reynolds number observed experimentally depends on both the spectrum and amplitude of the initial two- and three-dimensional disturbances to the flow. Typically, the natural transition is observed at Reynolds numbers of 2000 for pipe flow, 1000 for plane Poiseuille flow and plane Couette flow.

Hydrodynamic stability theory is an attempt to explain the mechanism of the instability of laminar fluid motion, it is the main method used now to discuss the transition process, although the theory of bifurcation and the theory of chaos now come into fashion. Someone suspected that the Navier-Stokes equations may be inadequate to describe turbulence, but to date there is no compelling evidence that there is any defect in its physical arguments. The theory of hydrodynamic stability is based on Navier-Stokes equations, investigates the stability behaviour of laminar solution, and regards the condition of instability as the condition of onset of turbulence.

By discussing the disturbances varying periodically both in streamwise direction and transverse direction, Squire (1933)^[14] verified that in two-dimensional flow of viscous fluid a two-dimensional disturbance is always more unstable than a three-dimensional one. Thus the solution of Orr-Sommerfeld equation has become the main task of linearized theory of hydrodynamic stability of viscous fluid. The equation was derived by Orr (1907) and Sommerfeld (1908) separately, they supposed the stream function of the disturbance has a form as follows

$$\psi(x, y, t) = \phi(y) \exp[i\alpha(x - ct)] \quad (1.1)$$

where $c = c_r + ic_i$, its imaginary part denotes whether the disturbance will grow up ($c_i > 0$), damp out ($c_i < 0$), or be neutrally stable ($c_i = 0$). Such disturbances are so called Tollmien-Schlichting waves.

The first attempt of this theory was on the stability of plane Couette flow, but no critical Reynolds number could be obtained, raising a suspicion whether such theory can be successful in transition problem. In his Ph.D. dissertation Heisenberg (1924)^[15] overcame the mathematical difficulties, particularly those related to critical layer (where $u = c$), and showed that plane Poiseuille is unstable at sufficiently large Reynolds numbers. This is the first rational calculation in viscous stability of parallel shear flow. Tollmien (1929)^[16] showed that it is necessary to involve a curve profile of velocity ($d^2u/dy^2 \neq 0$) and take account of viscosity near the boundary and at the critical layer (where the propagating velocity is equal to local mean velocity). His prediction and the calculation of Schlichting gave first critical Reynolds number verified by experiments, and thus appear the name Tollmien-Schlichting wave. C.C. Lin (1945)^[17] successfully settled the

mathematical problem of the asymptotic solution of higher order differential equation with turning point when solving Orr-Sommerfeld equation, he presented neutral curves and critical Reynolds numbers of plane Poiseuille flow and boundary layer flow, and explained the dual effect of viscosity to the stability of laminar motion. Through the dissipation of the energy, the viscous force will damp out the disturbances; on the other hand, the viscous force can produce Reynolds stress which converts energy from the basic flow to the disturbance, thus causing the disturbance to grow up. By this time, the mathematical problem related to linear hydrodynamic stability theory has been successfully resolved. With the later development of computers, many numerical solutions have been found.

There are some inherent defect in the linear theory. First, its result is independent of the amplitude of the initial disturbances, the amplitude is an arbitrary constant in the derived eigenvalue problem; second, the unsteady disturbance will grow up or damp out continuously, which contradicts the observed phenomena; and the critical Reynolds numbers thus obtained are often larger than those observed in experiments, especially in those of natural transition. Moreover, the linear theory can not explain the instability of plane Couette flow and pipe Poiseuille flow, all the existing linear analysis can only give stable results.

Reynolds suggested in 1883 the significance of finite-amplitude disturbances to the stability of pipe flow, and conjectured that there is a threshold amplitude of the disturbance, and when this value is exceeded, instability will occur. It could be said, however, the foundation of the theory of nonlinear hydrodynamic stability was laid by Landau in 1944^[18]. Landau suggested an equation now called weak nonlinear theory of the amplitude A of the fundamental mode of the disturbances

$$\frac{d|A|^2}{dt} = 2\sigma|A|^2 - l|A|^4 \quad (1.2)$$

l is so called Landau constant, when $l=0$, the equation is equivalent to that of linear theory; when $l \neq 0$, the second term at the right-hand side of the equation involves nonlinear effect. Letting the right-hand side of (1.2) be equal to zero, we can obtain the critical value A_{cr} of the amplitude. When $\sigma < 0$, $l < 0$, A_{cr} is called threshold amplitude, above which the disturbance grows, which is known as subcritical instability; when $\sigma > 0$, $l > 0$, A_{cr} is called equilibrium amplitude, the disturbance will approach this value, which is known as supercritical stability. The relationship of A and A_{cr} in these two cases is shown in Fig. 4.

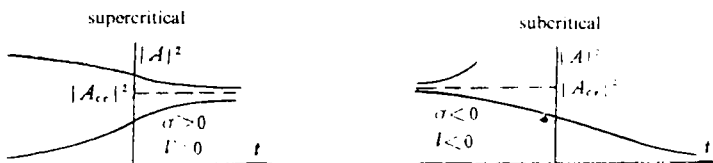


Fig. 4 Critical value of disturbance amplitude

Meksyn and Stuart (1951)^[19] discussed the interaction through Reynolds stress of mean flow and disturbances in plane Poiseuille flow. They determined the mean flow and the disturbances by solving the differential equation of mean velocity and Orr-Sommerfeld equation simultaneously, and obtained a relationship between threshold amplitude of disturbances and the critical Reynolds numbers. This method does not consider the generation and the interaction of harmonic

components. Landau predicted the behaviour of the nonlinear development of the disturbances by his equation, but he did not show how to derive this equation. The derivation of Landau equation, from hydrodynamic partial differential equations were done by Palm^[20] (1960, for Bénard problem), Stuart and Watson^[21] (1960, for parallel flow). What Stuart and Watson suggested is an asymptotic method based on Fourier analysis and taking the amplitude as the small parameter, which is generally accepted as a weak nonlinear theory which describes the hydrodynamic stability of the cases not very far from the neutral situation. The method involves the generation of harmonic components and the distortion of mean flow and fundamental disturbance owing to the interaction of the harmonic components. Since 1960, the Landau equations of many flows have been derived in succession.

Orszag and Patera (1983)^[22] suggested a concept of secondary instability. They through the instability process involves essentially three steps: (i) primary (linear) instability of the basic shear flow; (ii) nonlinear saturation of the primary instability and formation of a secondary flow; (iii) secondary instability, i.e. linear stability of the secondary flow. Classical hydrodynamic stability theory of C.C.Lin describes step (i), while the standard techniques of nonlinear stability theory of Stuart and Watson describe step (ii). Orszag and Patera assumed that the primary instability is two-dimensional, such unstable perturbations are saturated into an equilibrium, and an infinitesimal three-dimensional disturbance can cause strong instability to the equilibrium. The characteristic time scale of the instability is like that observed in experiments. Herbert (1983)^[23] suggested a similar subharmonic secondary instability theory. The three-dimensional disturbance he discussed has a wavelength as twice as that of two-dimensional equilibrium.

In three basic types of motion of the parallel shear flow, only plane Poiseuille flow has a critical Reynolds number $Re = 5772.2$ by linear stability analysis, but plane Couette flow and pipe Poiseuille flow are always linearly stable. All the techniques of nonlinear stability theory (including secondary instability theory) are based on the linear stability analysis, so the results with respect to plane Poiseuille flow of various theories are rather complete and consistent with each other. For plane Couette flow and pipe Poiseuille flow, however, there are only some results under certain assumptions, causing the controversial conclusions. For example, Davey and Nguyen (1971)^[24] and Itoh (1977)^[25] gave opposite results with respect to axisymmetrical finite-amplitude disturbances in pipe Poiseuille flow.

Bifurcation theory attempts to classify and characterize all solutions which can arise from the instability of a given solution when some parameters change in nonlinear system, so it is advantageous to use bifurcation theory in hydrodynamic stability problem. Now it is often used in Couette-Taylor flow.

The three basic motion in parallel shear flow can be explained with the concept of bifurcation theory as three types of bifurcation.

In other words, the positions of the intersection of bifurcation solution and basic solution have three situations as in Fig. 5 in conception.

Nonlinear theory of hydrodynamic stability is a perturbation method based on linear theory. The critical point obtained by linear theory is actually the bifurcation point, and what nonlinear theory wants to find is the real critical position of the bifurcation solution. When the position of bifurcation and actual critical position are far away, the theory will have difficulty, so it was called weak nonlinear theory. There is an assumption in this idea, namely the difference of these two positions is induced by nonlinear effect. Many nonlinear methods treating the plane Poiseuille flow

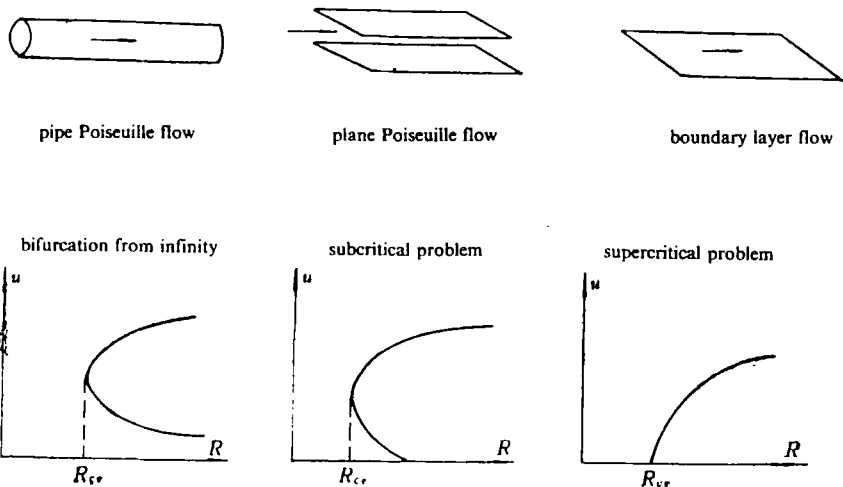


Fig. 5

successfully can not reach the results expected in pipe Poiseuille flow, showing the essential difference of the bifurcation from infinity and the subcritical problem. Many hydrodynamists, however, regard these two problems as one—subcritical problem.

In 1979, Rosenblat and Davis^[26] suggested the concept of the bifurcation from infinity, they pointed out through the analysis of a simple model equation that the bifurcation solution bifurcated from infinity and the eigensolution of the linearized equation are functions of two different kinds.

For one-dimensional nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{\mu} \frac{\partial^2 u}{\partial x^2} + u^2 - u^3 \quad 0 < x < 1 \quad (1.3)$$

$$u(0, t) = u(1, t) = 0 \quad (1.4)$$

the linearized form is

$$\frac{\partial u}{\partial t} = \frac{1}{\mu} \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \quad (1.5)$$

$$u(0, t) = u(1, t) = 0 \quad (1.6)$$

and the solution of which is

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp \left[-\frac{n^2 \pi^2 t}{\mu} \right] \sin(n\pi x) \quad (1.7)$$

If we look for a steady solution of the nonlinear equation (1.3), letting

$$\varepsilon = 1/\mu \quad (1.8)$$

$$u = \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \dots \quad (1.9)$$

then the equations of u_1, u_2 are

$$u_1'' + u_1^2 = 0, \quad u_1(0) = u_1(1) = 0 \quad (1.10)$$

$$u_2'' + 2u_1 u_2 = u_1^3, \quad u_2(0) = u_2(1) = 0 \quad (1.11)$$

We can see clearly that the eigenfunctions of the linearized equation are trigonometric functions, and the bifurcation solutions bifurcated from infinity of the nonlinear equation are elliptic functions. These are two quite different functions. In the techniques of weak nonlinear stability theory, it is the eigenfunctions of the linearized equation that are taken as the basis of the equilibrium solution of nonlinear disturbances, so this technique is not suitable to the problem of the bifurcation from infinity.

H. Zhou (1982)^[27] improved the techniques of the weak nonlinear stability theory. He set the starting point of the perturbation method on the laminar velocity profile that was distorted by the nonlinear effect, and obtained better results in plane Poiseuille flow. From the point of view of bifurcation theory, we can say that Zhou's method is a method of perturbed bifurcation theory. The fundamental mode of the disturbance in his method is no longer the eigenfunctions of original Orr-Sommerfeld equation.

Gill (1962)^[28] discussed the instability mechanism of pipe Poiseuille flow on the basis of Leite's experiment. He thought that the main cause of the instability was those disturbances which change the mean velocity slightly but the curvature finitely.

We can imagine that some modification of mean velocity has certain significance to the stability behaviour of fluid motion. If we introduce some reasonable distortion profile, even in the most difficult problem of the bifurcation from infinity, we can obtain some linearized eigenfunction close to the eigenfunction of the original problem and arrive at better result. To find such distortion profiles, we can draw support from the concept of renormalization in quantum field theory^[29].

When applying quantum field theory to some physical process, the results of the first order approximation are quite close to those of experiments. But if we want to find more exact results and take the higher order approximation, we always obtain infinity. Take the quantum electrodynamics as an example, the infinity in the calculation only happens in two physical aspects: One is the change of the self-energy or the static mass of the electron; the other is the change of the electric charge of the electron, or the interaction constant of the electron and the electromagnetic field. The mass of the electron observed in experiments is the whole mass of the electron. It is impossible to tell in experiments that which part of the mass is the "inherent" mass of the electron, which part is the electromagnetic mass of the electron, or the self-energy produced by the interaction with the electromagnetic field. So, in theory we should take the "inherent" mass and the electromagnetic mass of the electron together as the mass of the electron observed in experiments. It is also impossible to tell in experiments which part of the charge of the electron is the "inherent" charge of the electron, and which part is the additive charge of the electron produced by vacuum polarization. So in theory we should also take the "inherent" charge and the additive charge produced by vacuum polarization together as the charge of the electron observed in experiments. Such redefinition of the mass of the electron is called mass renormalization of the electron, and the redefinition of the charge is called the charge renormalization of the electron. After the renormalization of the charge and the mass, the infinity in quantum electrodynamics has been absorbed in the charge and the mass. The calculation of the higher order approximation of the interaction between the electron and the electromagnetic field can be carried out. The suggestion of the method of the renormalization brings out great achievements in quantum field theory.

What the hydrodynamic stability theory wants to investigate is the problem of the interaction between disturbances and the basic laminar flow. It is well known that disturbances distort the basic laminar flow continuously in its development due to the nonlinear effect. So the laminar flow consists of the basic flow and the modification by the disturbances throughout the whole development of the disturbances. In the laboratory, we can observe basic laminar flow at first, and this profile is distorted after the introduction of disturbances. If the flow turns into turbulence, we can observe the mean velocity profile of fully developed turbulent flow. In this process of transition, the mean velocity profile is changing continuously. Take the experience of the conception of renormalization in quantum field theory, we should also take the basic laminar flow and the modification produced by disturbances as a whole, and as the real mean velocity profile to study its stability behaviour.

As in quantum field theory, the part introduced by renormalization reflects the interaction between the electron and the electromagnetic field, the distortion velocity profile in hydrodynamic stability theory should reflect the interaction between basic laminar flow and the disturbances. In section 2, we first study the possible types of distortion profiles of mean velocity, obtain a kind of distortion profile which varies with time, and has no transverse components and no mixing effect in the flow, then we suggest a hydrodynamic stability theory of distorted laminar flow. In the following two parts of this study (II, III that will be published later), we use such distortion profiles to analyse the stability behaviour of three basic motions in parallel shear flow, thus we suggest a possible new approach which leads to instability.

II. A Kind of Distortion Profile of Mean Velocity in Parallel Shear Flow

In the analysis of hydrodynamic stability, we often suppose that the disturbance stream function has a form as follows

$$\psi = \phi(y) \exp[i\alpha(x - ct)] + C.C. \quad (2.1)$$

where α is a real number, C.C. is complex conjugate of the first term, c is a complex number, $c = c_r + ic_i$, when $c_i > 0$, the disturbance concerned grows up with time, the laminar flow is unstable; when $c_i < 0$, the disturbance damps out with time, the laminar flow is stable.

Because of the quadratic nonlinearity of the convective term in hydrodynamic equation, the solution of the perturbation method, which is always used to analyse nonlinear equation, will involve higher harmonic components and the zero wave number component

$$\Phi(y) \exp[2\alpha c_i t] \quad (2.2)$$

This is just the part which modifies the basic laminar flow. We look for such a solution directly in hydrodynamic equation, and obtain two distortion profiles in parallel shear flow as follows.

In axisymmetric pipe Poiseuille flow, we obtain a distortion profile which has no extra flux^[30]

$$u(r, t) = \frac{1}{r} \frac{\partial \psi}{\partial r} = \left\{ \sum_{n=0}^{\infty} \frac{(\mu R)^n r^{2n}}{[(2n)!!]^2} - A \sum_{n=0}^{\infty} \frac{(\mu R)^n r^{2n+2}}{[(2n+2)!!]^2} \right\} \exp[\mu t] \quad (2.3)$$

$$A = \sum_{n=0}^{\infty} \frac{(\mu R)^n}{[(2n)!!]^2} / \sum_{n=0}^{\infty} \frac{(\mu R)^n}{[(2n+2)!!]^2} \quad (2.4)$$

where $(2n)!! = 2 \cdot 4 \cdot 6 \cdots 2n$, when $n = 0$, $(2n)!! = 1$

μ is a real number, R is Reynolds number

$$\begin{aligned}\mu R = & -26.374614, -70.849999, -135.02071, -218.92019, \\ & -322.55512, -445.92756, -589.03835, -751.88837, \\ & -930.86419, \dots\end{aligned}$$

This distortion profile decays with time. Its shape is presented in paper [30]. It is very interesting that the number of inflection is the same as the sequence number of the constants μR . The positions of the inflections of the profiles are showed in table 1.

Table 1 The position of the inflections in distortion profiles in pipe Poiseuille flow

μR	r_{inflex}		
- 26.374614	0.36861230		
- 70.849999	0.21873950	0.63339529	
-135.02071	0.15845172	0.45882234	0.73463279
-218.92019	0.12443835 0.79116268	0.36033118	0.57693594
-322.55512	0.10261683 0.65178856	0.29686392 0.82760234	0.47530079
-445.92756	0.08718966 0.55434041	0.25247164 0.70386848	0.40423912 0.85312926
-589.03835	0.07586219 0.48232183 0.87203428	0.21967112 0.61242356	0.35172134 0.74229273
-751.88837	0.06714605 0.42690576 0.77184245	0.19443216 0.54206954 0.86680868	0.31131054 0.65700747
-930.86419	0.06043676 0.38367680 0.69368480	0.17474373 0.48718998 0.79682968	0.27978688 0.59047816 0.89993119

Now we look for such solutions directly in Navier-Stokes equation.

Still suppose

$$u = u(r) \exp[\mu t] \quad (2.5)$$

Substituting it into the Navier-Stokes equations expressed in cylindrical coordinates, we can obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{R} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad u(0,0) = 1, \quad u(1,t) = 0 \quad (2.6)$$

Substituting (2.3) into the equation, it satisfies the boundary condition; and

$$\frac{\partial p}{\partial x} = \frac{1}{R} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \mu R u \right) = -\frac{A}{R} \exp[\mu t] \quad (2.7)$$

A is represented by (2.4).

So it is necessary to have a disturbance pressure gradient to sustain the flux, the pressure gradient decays with time, and has an order of magnitude $O(1/R)$.

For two-dimensional plane parallel flow, we can introduce a stream function as follows

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \quad (2.8)$$

where u is the streamwise velocity, v is the transverse velocity.

The equations satisfied by the stream function are

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi - \frac{1}{R} \nabla^4 \psi = 0 \quad (2.9)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0, \quad \text{when } y = \pm 1 \quad (2.10)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We look for a solution which can vary with time, and have no transverse velocity, no mixing effect and extra flux.

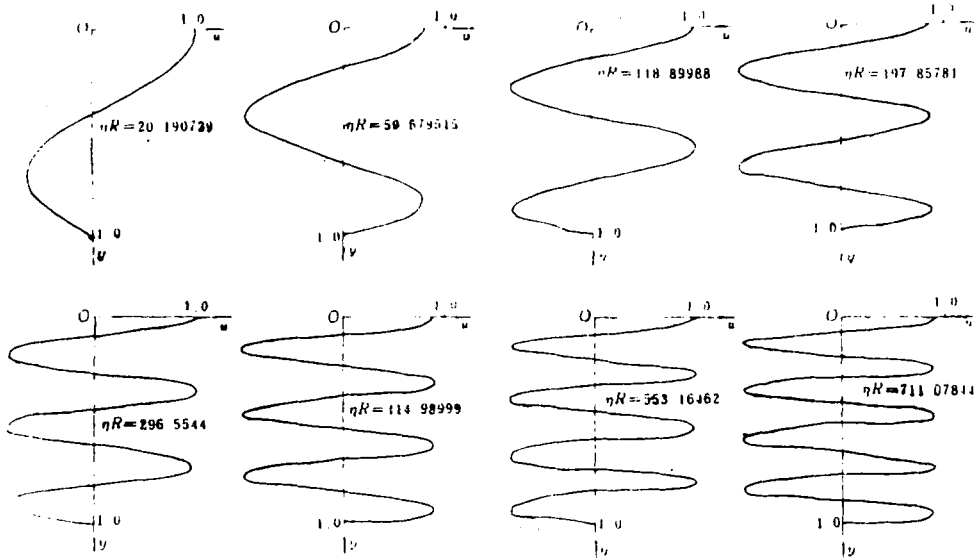


Fig. 6 Distortion profiles in planar parallel flow

Suppose

$$\psi = \phi(y) \exp[-\eta t], \quad \eta > 0 \quad (2.11)$$

Substituting it into (2.9), (2.10), we can obtain

$$\phi(4) + \eta R \phi'' = 0 \quad (2.12)$$

$$\phi(-1) = \phi(1) = \phi'(-1) = \phi'(1) = 0 \quad (2.13)$$

From (2.12), (2.13), we can obtain

$$\phi(y) = C \left[\cos(\sqrt{\eta R}) \cdot y - \frac{1}{\sqrt{\eta R}} \sin(\sqrt{\eta R}) y \right] \quad (2.14)$$

$$u = \frac{\partial \psi}{\partial y} = C [\cos(\sqrt{\eta R}) - \cos(\sqrt{\eta R}) y] \exp[-\eta t] \quad (2.15)$$

$$C = \begin{cases} -1/(\cos\sqrt{\eta R} + 1), & \cos\sqrt{\eta R} > 0 \\ 1/(\cos\sqrt{\eta R} - 1), & \cos\sqrt{\eta R} < 0 \end{cases} \quad (2.16)$$

The determination of the constant C is to make the distortion profile have nonzero streamwise velocity at $y=0$, and $|u_{max}|=1$, thus can define the distortion level of the laminar flow conveniently. In satisfying the condition of no extra flux, we obtain

$$\eta R = 20.190729, 59.679515, 118.89988, 197.85781, 296.5544, \\ 414.98999, 553.16462, 711.07844, 888.73143, \dots$$

Comparing with the values of μR of the distortion profiles in axisymmetrical pipe Poiseuille flow, we can see that the distortion profile sustains longer in plane parallel shear flow. The shapes of the distortion profile can be seen in Fig. 6. The number of the inflection is also the same as the sequence number of the constants ηR . The positions of the inflections of the distortion profiles are showed in table 2.

Table 2 The position of the inflections in distortion profiles in planar parallel flow

ηR	y_{inflex}		
20.190729	0.3495778		
59.679515	0.2033327	0.6099981	
118.89988	0.1440553	0.4321658	0.7202764
197.85781	0.1116717	0.3350152	0.5583687
	0.7817022		
296.5544	0.0912153	0.2736459	0.4580765
	0.6385071	0.8209379	
414.98999	0.0771083	0.2313249	0.3856415
	0.5397580	0.6939746	0.8481912
553.16462	0.0667871	0.2003613	0.3339356
	0.4675098	0.6010840	0.7346583
	0.8682325		
711.07844	0.0589062	0.1767187	0.2945311
	0.4123435	0.5301559	0.6479684
	0.7657808	0.8835932	

We also suppose

$$u = \mu(y) \exp[-\eta t], \quad \eta > 0 \quad (2.17)$$

Substituting it into the Navier-Stokes equations, we can obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{R} \frac{\partial^2 u}{\partial y^2}, \quad u(\pm 1, t) = 0 \quad (2.18)$$

From (2.15)

$$\frac{\partial p}{\partial x} = C\eta \cos(\sqrt{\eta R}) \exp[-\eta t] \quad (2.19)$$

C is represented by (2.16). The disturbance pressure gradient also decays with time, and its order of magnitude is $O(1/R)$.

It is necessary to point out that the condition of the zero flux is a condition to determine the integral constants, and the other approach can be obtained by set $\partial p / \partial x = 0$

In this case, from (2.7), (2.3) becomes

$$u(r, t) = \sum_{n=0}^{\infty} \frac{(\mu R)^n r^{2n}}{[(2n)!!]^2} \exp[\mu t] \quad (2.20)$$

and $\mu R = -\gamma^2$, γ is the zero points of Bessel function J_0 . This is just the so called "free mode" indicated by Davy and Nguyen (1971)^[24]. In solving the equations of nonlinear hydrodynamic stability theory, they found such mode is the solution of the corresponding homogeneous equation of the perturbation equations determining the fundamental disturbance. Because it is independent

Table 3 The position of the inflections in "free modes" in pipe Poiseuille flow

μR	r_{inflex}		
- 30.471504	0.33354111	0.96582413	
- 74.886524	0.21276261	0.61608824	0.98643544
- 139.03947	0.16614548 0.99274979	0.45214310	0.72393875
- 222.93177	0.12331314 0.78401260	0.36707474 0.99549179	0.57172165
- 326.56466	0.10188522 0.64777451	0.29502640 0.82250595	0.47237320 0.99692459
- 449.93197	0.08680038 0.55186844 0.99777298	0.25134506 0.70072861	0.40243607 0.84932480

Table 4 The position of the inflections in "free modes" in planar parallel flow

ηR	y_{inflex}		
22.206610	0.3333337		
61.685029	0.2	0.6	
120.90265	0.1428571	0.4285714	0.7142857
199.85949	0.1111111 0.7777777	0.3333333	0.5555556
298.55556	0.0909091 0.6363636	0.2727273 0.8181818	0.4545455
416.99078	0.0769231 0.5384615	0.2307692 0.6923077	0.3846154 0.8461539
555.16525	0.0666667 0.4666667 0.8666667	0.2 0.6	0.3333333 0.7333333
713.07694	0.0588235 0.4117647 0.7647059	0.4764706 0.5294418 0.8823529	0.2941177 0.6470588

of the right-hand terms of the equations, Herbert^[3] called it "free mode". Such modes are also the solutions of Navier-Stokes equations, and also have many inflections in the profile, and the positions are showed in table 3.

In the same way, when the pressure gradient vanishes, (2.15) is still

$$u = C[\cos(\sqrt{\eta R} x) - \cos(\sqrt{\eta R} y)] \exp[-\eta t] \quad (2.21)$$

$$\text{but } \sqrt{\eta R} = (\pi/2)(2n+1), \quad n=0, 1, 2, \dots \quad (2.22)$$

This is just the "free mode" pointed out by Herbert (1982)^[3] and the positions of the inflections in the profiles are showed in table 4.

So, we can regard (2.3) and (2.15) as the general expression of this kind of distortion profile which varies with time, and has no transverse components and no mixing effect. When we use different method to determine the integral constants, the constant C in the expression and the values of μR (or ηR) are also different, as well as the positions of the inflections. But we can see later that the influence on the stability behaviour of the flow is quite similar.

III. The Hydrodynamic Stability Theory of Distorted Laminar Flow

Now, we define the problem of hydrodynamic stability of distorted laminar flow mathematically.

In certain region V in the space with the boundary ∂V , the Navier-Stokes equations are

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \nabla \bar{u} \right) = -\nabla p + \mu \nabla^2 \bar{u} + \bar{G}(\bar{x}, t) \quad \bar{x} \in V \quad (3.1)$$

$\bar{u}(\bar{x}, t)$ is the velocity field, p is the pressure field, ρ is the density of the fluid, μ is viscosity coefficient, $\bar{G}(\bar{x}, t)$ is prescribed body force field. And \bar{u} satisfies

$$\nabla \cdot \bar{u} = 0 \quad \bar{x} \in V \quad (3.2)$$

and the boundary conditions

$$\bar{u}(\bar{x}, t) = \bar{u}_B(\bar{x}, t) \quad \bar{x} \in \partial V \quad (3.3)$$

Suppose (\bar{u}, p) and (\bar{u}_t, p_t) are two laminar solutions which satisfy Navier-Stokes equations respectively, where (\bar{u}, p) is basic laminar solution, and both can exist in the flow simultaneously provided

$$\bar{u} \nabla \bar{u}_t + \bar{u}_t \nabla \bar{u} = 0 \quad \bar{x} \in V \quad (3.4)$$

Suppose another solution (\bar{u}^*, p^*) also satisfies Navier-Stokes equations, the three of them have different initial conditions, the difference $(\bar{u}^* - \bar{u} - \bar{u}_t, p^* - p - p_t) = (\bar{u}', p')$ satisfies the equations

$$\left. \begin{aligned} \rho \left(\frac{\partial \bar{u}'}{\partial t} + \bar{u}_t \nabla \bar{u}' + \bar{u} \nabla \bar{u}' + \bar{u}' \nabla \bar{u}_t + \bar{u}' \nabla \bar{u} + \bar{u}' \nabla \bar{u}' \right) &= -\nabla p' + \mu \nabla^2 \bar{u}' \quad \bar{x} \in V \\ \nabla \cdot \bar{u}' &= 0 \end{aligned} \right\} \quad (3.5)$$

$$\text{with boundary condition} \quad \bar{u}'(\bar{x}, t) = 0 \quad \bar{x} \in \partial V \quad (3.6)$$

$$\text{and initial condition} \quad \bar{u}'(\bar{x}, 0) = \bar{u}'_0 \quad \bar{x} \in V \quad (3.7)$$

The problem of the hydrodynamic stability of distorted laminar flow is to solve (3.5) under conditions of (3.6) and (3.7). If the amplitude of the disturbance \bar{u}' grows up with time or in space, the distorted laminar flow is unstable, if the amplitude of the disturbance \bar{u}' damps out with time or in space, the distorted laminar flow is stable.

$$\text{Supposing} \quad \bar{u}^* = \bar{u} + \bar{u}_1 + \varepsilon \bar{u}' \quad (3.8)$$

$$p^* = p + p_1 + \varepsilon p' \quad (3.9)$$

and neglecting the higher order of ε , we can obtain

$$\left. \begin{aligned} \rho \left(\frac{\partial \bar{u}'}{\partial t} + \bar{u}_1 \nabla \bar{u}' + \bar{u} \nabla \bar{u}' + \bar{u}' \nabla \bar{u}_1 + \bar{u}' \nabla \bar{u} \right) &= -\nabla p' + \mu \nabla^2 \bar{u}' \quad \bar{x} \in V \\ \nabla \cdot \bar{u}' &= 0 \end{aligned} \right\} \quad (3.10)$$

The problem of the linear hydrodynamic stability of distorted laminar flow is to solve (3.10) under the conditions of (3.6) and (3.7). If the amplitude of the disturbance \bar{u}' grows up with time or in space, the distorted laminar flow is linearly unstable, if the amplitude of the disturbance \bar{u}' damps out with time or in space, the distorted laminar flow is linearly stable. Here we suggest a new idea. Following the classic theory of hydrodynamic stability, if we introduce a disturbance into the basic laminar flow, when the disturbance grows up with time or in space, the flow may convert into turbulence (as in plane Poiseuille flow), or into another stable laminar flow (as the Taylor toroidal vortex in the flow between two rotating concentric cylinders). What we suggest is that there can be two laminar flows existing in the flow field simultaneously, one of which is the steady basic laminar flow, the other is of another kind. If we introduce a disturbance into the compound flow consisting of the two laminar flows, or the basic laminar flow distorted by the second flow (which is usually a small quantity), when the disturbance grows up with time or in space, the flow may convert into turbulence or another laminar flow. This conception is rather different from that of Orszag et al. They studied the development of the disturbance in the basic laminar flow. The first stage is based on the linear analysis of classic stability theory, and the second laminar flow they introduced, the so called equilibrium, is an approximate solution of classic linear stability equation, not the exact solution of Navier-Stokes equations. We study the hydrodynamic stability of distorted laminar flow. The linear analysis is different, and the mean velocity always involves the basic laminar flow and the second laminar flow induced by some effects.

Taylor (1936)^[32] raised doubts about the work of Tollmien. In the laboratory, the instability of laminar boundary layer depends on the turbulent perturbations in main stream, the critical Reynolds number in pipe Poiseuille flow is also decided by how smoothly the fluid runs into the pipe. So Taylor thought the critical Reynolds numbers only relate to the turbulence level in the stream. He suggested that the turbulence disturb the boundary layer through the local pressure gradient related to turbulent pressure. If the gradient is efficiently large and has an opposite direction against the stream, the boundary layer will become turbulent. Taylor derived out some relationship between local pressure gradient and the scales in turbulence and obtained some curves reflecting the relationship of critical Reynolds numbers and the turbulence level by experiments. Schubauer and Skramstad^[33] confirmed the existence of Tollmien-Schlichting wave in their experiments of plate boundary layer in 1940s, thus affirmed the hydrodynamic stability theory. But

the turbulent noise has never been introduced into the hydrodynamic stability theory so far. According to the experiments mentioned above, the background turbulence noise does have certain effects upon the stability behaviour of the flow.

The distortion profile we gave in section 2 is just the second laminar solution which can exist in the flow field with the basic laminar flow simultaneously, and it reflects the interaction between the disturbances and the basic laminar flow, and this distortion profile can also be regraded as a representation of the effect on the mean velocity by turbulent Reynolds stress. So the distortions we introduce as the second laminar flow is to some extent like the assumption suggested by Taylor, reflecting a kind of influence of background turbulence noise. We have not, however, included all the influence of background noise. We only consider the modification of mean velocity. The pressure gradient related to the distortion is not definitely opposite to the flow direction, but it may also induce instability of the flow.

In the later two parts of this paper, we shall use the distortion profile to discuss the application of hydrodynamic stability theory of distorted laminar flow in particular flows.

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