

# BACKLUND TRANSFORMATIONS FOR THE EQUATION

$$\partial^2 u / \partial x^1 \partial x^1 + \partial^2 u / \partial x^2 \partial x^2 = f(u)$$

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## Abstract

*Bäcklund transformations for the equation  $\partial^2 u / \partial x^1 \partial x^1 + \partial^2 u / \partial x^2 \partial x^2 = f(u)$  (here  $f$  is an arbitrary function) is studied in this paper, using the procedure of Wahlquist and Estabrook (WEP). We conclude that the condition  $d^2 f / du^2 = \lambda f$  is sufficient for the existence of Bäcklund transformations for the equation of our interest. A special case of our results leads to the conclusion of Leibbrandt<sup>[1,2]</sup>.*

## I. Introduction

If we consider the steady two-dimensional Euler-fluid-flow problem, then the governing equations are as follows

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (1.1)$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (1.2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (1.3)$$

where  $(x, y)$ ,  $(v_x, v_y)$  are spatial coordinates and velocity of fluid particle respectively,  $p$  is pressure,  $\rho$  is density. We suppose that  $v_x = \partial \phi / \partial y$ ,  $v_y = -\partial \phi / \partial x$  ( $\phi$  is Stokes stream function.) and eliminate the pressure term; then equations (1.1)–(1.3) are replaced by one equation

$$v_x \frac{\partial \Delta \phi}{\partial x} + v_y \frac{\partial \Delta \phi}{\partial y} = 0 \quad (1.4)$$

where  $\Delta \phi = (\partial^2 / \partial x^2 + \partial^2 / \partial y^2) \phi$ ,  $-\Delta \phi = \Omega$  (vorticity). The physical meaning of (1.4) is that vorticity is only dependent on  $\phi$ . Conversely, if vorticity is only dependent on  $\phi$ , then equation (1.4) is valid. Thus equation (1.4) is equivalent to the following equation

$$\Delta \phi = f(\phi) \quad (1.5)$$

where  $f(\phi)$  is an arbitrary function of  $\phi$ . Equation (1.5) is just the equation discussed in this paper. A special case of (1.5), i.e. the elliptic sine equation  $\Delta \phi = \sin \phi$ , is closely related to the problem of magnetic flux through a long Josephson junction.<sup>[1,2]</sup> The long time behavior of solutions of two-dimensional Euler equations have been studied by many authors<sup>[4]</sup>. It has applications in geophysics and plasmas<sup>[4]</sup>. But it is still an open problem up to now<sup>[4]</sup>.

## II. Wahlquist-Estabrook Procedure (WEP)<sup>[3]</sup>

We briefly outline the main results of WEP. For details see [3]. Let  $M, N, N'$  ( $M = \mathbf{R}^m$ ,  $N = \mathbf{R}^n$ ,  $N' = \mathbf{R}^n$ ) denote the spaces of independent variables, dependent variables and new dependent variables respectively,  $J^k(M, N)$  denote the  $k$ -jet bundle of maps from  $M$  to  $N$ ,  $\Sigma^k$  denote an exterior differential system of  $m$ -forms on  $J^k(M, N)$  associated with a quasilinear system  $R^{k+1} \subset J^{k+1}(M, N)$ . We choose coordinates  $x'^a$ ,  $y'^\mu$ , and  $y'^\mu$  ( $a = 1, \dots, m$ ;  $\mu = 1, \dots, n$ ) on  $J^1(M, N')$ . Then a basis for the contact  $(m-1)$ -forms on  $J^1(M, N')$  is given by  $\theta'^\mu \wedge w'_{ab}$ , where

$$\theta'^\mu := dy'^\mu - y'^\mu_a dx'^a \quad (2.1)$$

and for  $a < b = 1, 2, \dots, m$ ;  $w'_{ab}$  is the  $(m-2)$ -form defined by

$$w'_{ab} = \frac{\partial}{\partial x'^b} \lrcorner \frac{\partial}{\partial x'^a} \lrcorner (dx'^1 \wedge \dots \wedge dx'^m) \quad (2.2)$$

Let  $\Omega'$  denote the module of contact  $(m-1)$ -forms on  $J^1(M, N)$ . If we suppose now that the map  $\psi: J^k(M, N) \times_M J^0(M, N') \rightarrow J^1(M, N')$  is given in coordinates by

$$x'^a = x^a, \quad y'^\mu = y^\mu, \quad y'^\mu = \psi^*_a(x^b, z^A, \dots, z^A_{a_1} \dots a_k, y^s) \quad (2.3)$$

where  $x^a$  ( $a = 1, \dots, m$ ),  $z^A, y^s$  ( $A, \mu = 1, \dots, n$ ) are coordinates on  $M, N$  and  $N'$  respectively,  $J^k(M, N) \times_M J^0(M, N')$  is the fibered product of two jet bundles  $J^k(M, N)$  and  $J^0(M, N')$ ; then the requirement

$$\psi^* d\Omega' \subset \mathcal{I}(\Sigma^k, \psi^* \Omega') \quad (2.4)$$

implies that  $\psi$  is a Bäcklund map for  $R^{k+1}$ . Here  $\psi^*$  is the pull-back map associated with  $\psi$ ,  $d$  denotes exterior differential,  $\mathcal{I}(\Sigma^k, \psi^* \Omega')$  denotes the exterior ideal generated by  $\Sigma^k$  and  $\psi^* \Omega'$ . If  $\psi$  is a Bäcklund map for  $R^{k+1}$ , then  $\psi$  is a Bäcklund transformation provided that the image of the map  $\psi^r$  ( $r$ -th prolongation of  $\psi$ ) is contained in a system of differential equations  $R^1$ .

## III. Formulation of the Bäcklund Transformations for the Equation $\left( \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} \right) u = f(u)$

We now consider the following equation

$$\left( \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} \right) u = f(u) \quad (3.1)$$

where  $x^1, x^2$  are independent variables and  $u$  is the dependent variable,  $f$  is an arbitrary function of  $u$ ,  $M = \mathbf{R}^2$  (with coordinates  $x^1, x^2$ ),  $N = \mathbf{R}^1$  (with coordinate  $u$ ),  $N' = \mathbf{R}^1$  (with coordinate  $v$ ). Let  $w$  denote the volume form on  $M$ ,  $w = dx^1 \wedge dx^2$ ,  $\theta$  denote contact 1-form on  $J^1(M, N)$ ,  $\theta = du - u_1 dx^1 - u_2 dx^2$ . The exterior differential system of 2-forms on  $J^1(M, N)$  associated with (3.1) is generated by

$$\sigma := du_1 \wedge dx^2 - du_2 \wedge dx^1 - f w \quad (3.2)$$

$$\eta_1 := \theta \wedge w_1 = du \wedge dx^2 - u_1 dx^1 \wedge dx^2 \quad (3.3)$$

$$\eta_2 := \theta \wedge w_2 = du \wedge dx^1 + u_2 dx^1 \wedge dx^2 \quad (3.4)$$

We seek the following form of Bäcklund maps for equation (3.1)

$$x'^i = x^i \quad (i=1, 2), \quad v' = v, \quad v'_a = \psi_a(u, u_1, u_2, v) \quad (a=1, 2) \quad (3.5)$$

A basis for the contact 1-forms on  $J^1(M, N')$  is given by

$$\theta' \wedge w_{12} = dv - \psi_1 dx^1 - \psi_2 dx^2 \quad (\text{it has been pulled back}) \quad (3.6)$$

Then equation (2.4) requires that

$$d\psi_1 \wedge dx^1 + d\psi_2 \wedge dx^2 = f_1 \eta_1 + f_2 \eta_2 + g\sigma + \xi \wedge \xi \quad (3.7)$$

here  $f_1, f_2, g$  are arbitrary functions on  $J^1(M, N) \times_{\mathbf{M}} J^0(M, N')$  and  $\xi = dv - \psi_1 dx^1 - \psi_2 dx^2$ ,  $\xi$  is a 1-form on  $J^1(M, N) \times_{\mathbf{M}} J^0(M, N')$ . Note that

$$\frac{\partial \psi_a}{\partial v} dv = \frac{\partial \psi_a}{\partial v} \xi + \psi_1 \frac{\partial \psi_a}{\partial v} dx^1 + \psi_2 \frac{\partial \psi_a}{\partial v} dx^2 \quad (a=1, 2)$$

then it is easily shown, from equation (3.7), that

$$\left. \begin{aligned} \frac{\partial \psi_1}{\partial u} &= f_2, \quad \frac{\partial \psi_1}{\partial u_1} = 0, \quad \frac{\partial \psi_2}{\partial u} = f_1, \quad \frac{\partial \psi_2}{\partial u_2} = 0 \\ \frac{\partial \psi_1}{\partial u_2} &= -g = -\frac{\partial \psi_2}{\partial u_1} \\ \psi_1 \frac{\partial \psi_2}{\partial v} - \psi_2 \frac{\partial \psi_1}{\partial v} &= -f_1 u_1 + f_2 u_2 - gf \end{aligned} \right\} \quad (3.8a \sim f)$$

$(\partial/\partial u_1)(3.8e)$  and  $(\partial/\partial u_2)(3.8e)$  show that  $g = g(u, v)$ , then from (3.8b,d,e) we have

$$\psi_1 = -g(u, v)u_2 + h_1(u, v) \quad (3.9)$$

$$\psi_2 = g(u, v)u_1 + h_2(u, v) \quad (3.10)$$

Substitutions of (3.9), (3.10) into (3.8) show that

$$\frac{\partial g}{\partial u} = 0 \quad (3.11)$$

$$h_1 \frac{\partial g}{\partial v} - g \frac{\partial h_2}{\partial v} + \frac{\partial h_2}{\partial u} = 0 \quad (3.12)$$

$$h_2 \frac{\partial g}{\partial v} - g \frac{\partial h_1}{\partial v} - \frac{\partial h_1}{\partial u} = 0 \quad (3.13)$$

$$h_1 \frac{\partial h_2}{\partial v} - h_2 \frac{\partial h_1}{\partial v} + fg = 0 \quad (3.14)$$

i.e.

$$g = g(v) \quad (3.15)$$

$$\frac{\partial}{\partial u} \left( \frac{h_2}{g} \right) = \frac{\partial}{\partial v^*} \left( \frac{h_1}{g} \right) \quad (3.16)$$

$$\frac{\partial}{\partial u} \left( \frac{h_1}{g} \right) = -\frac{\partial}{\partial v^*} \left( \frac{h_2}{g} \right) \quad (3.17)$$

$$f = \frac{1}{2} \frac{\partial}{\partial u} \left[ \left( \frac{h_1}{g} \right)^2 + \left( \frac{h_2}{g} \right)^2 \right] \quad (3.18)$$

Here,  $dv^*=dv/g$

It can be verified that if conditions (3.15)–(3.18) are satisfied, then equation

$$u_{11} + u_{22} = f(u) \quad (3.19)$$

is transformed into

$$v_{11} + v_{22} - \frac{g'}{g} [v_1^2 + v_2^2 - (h_1^2 + h_2^2)] - \frac{1}{2} \frac{\partial}{\partial v} (h_1^2 + h_2^2) = 0 \quad (3.20)$$

under the Bäcklund transformations

$$\psi_1 = -g(v)u_2 + h_1(u, v) \quad (3.21)$$

$$\psi_2 = g(v)u_1 + h_2(u, v) \quad (3.22)$$

Now we require that  $(\partial/\partial v)[(h_1^2 + h_2^2)/g^2] = M^*(v)$ ,  $M^*$  is an arbitrary function, so that equation (3.20) is concerned with only one dependent variable. Thus

$$h_1^2 + h_2^2 = g^2(M(v) + N(u)), \quad M(v) = \int M^*(v) dv \quad (3.23)$$

where  $N$  is an arbitrary function. From (3.18), we have

$$f = \frac{1}{2} N' \quad (3.24)$$

Thus

$$N = 2 \int f du \quad (3.25)$$

Table 1 Bäcklund transformations for various solutions of (3.30)

$f$	Defining equation for $u$	Bäcklund transformations	Defining equation for $v$
1. $\omega^2 u$ ( $\omega$ is a real const.)	$u_{11} + u_{22} = \omega^2 u$	$v_1 = -u_2 + \omega \sqrt{u^2 + v^2} \cos \left\{ \theta + \tan^{-1} \frac{u}{v} \right\}$ $v_2 = u_1 + \omega \sqrt{u^2 + v^2} \sin \left\{ \theta + \tan^{-1} \frac{u}{v} \right\}$ $\theta$ is a Bäcklund parameter	$v_{11} + v_{22} = \omega^2 v$
2. $\omega \alpha^2 (1 + \gamma^2) (e^{2\omega u} - \beta^2 e^{-2\omega u})$ , $\omega, \alpha, \gamma, \beta$ are constants.	$u_{11} + u_{22} = \omega \alpha^2 (1 + \gamma^2) \cdot (e^{2\omega u} - \beta^2 e^{-2\omega u})$	$v_1 = -u_2 + \alpha (e^{\omega u} + \beta e^{-\omega u}) (\cos \omega v + \gamma \sin \omega v)$ $v_2 = u_1 - \alpha (e^{\omega u} - \beta e^{-\omega u}) (\sin \omega v - \gamma \cos \omega v)$	$v_{11} + v_{22} = 2\omega \alpha^2 \beta^2 (e^{2\omega v} - 1) \sin 2\omega v + 2\gamma \cos 2\omega v]$
3. Special case of 2 $\beta = 0$	$u_{11} + u_{22} = k e^{2\omega u}$ $k = \omega \alpha^2 (1 + \gamma^2)$ elliptic Liouville equation	$v_1 = -u_2 + \alpha e^{\omega u} (\cos \omega v + \gamma \sin \omega v)$ $v_2 = u_1 - \alpha e^{\omega u} (\sin \omega v - \gamma \cos \omega v)$	$v_{11} + v_{22} = 0$ Laplace equation
4. Special case of 2 $\omega = 1/2, \beta = -1, \gamma = 0, \alpha = 1$ . Leibbrandt's cases <sup>[1,2]</sup>	$u_{11} + u_{22} = \sinh u$	$v_1 = -u_2 + 2 \sinh(u/2) \cos(v/2)$ $v_2 = u_1 - 2 \cosh(u/2) \sin(v/2)$	$v_{11} + v_{22} = \sin v$
5. Special case of 2 $\omega = 1/2, \gamma = 0, \alpha = \beta = 1, f = \sinh u$	$u_{11} + u_{22} = \sinh u$	$v_1 = -u_2 + 2 \cosh(u/2) \cos(v/2)$ $v_2 = u_1 - 2 \sinh(u/2) \sin(v/2)$	$v_{11} + v_{22} = -\sin v$

From (3.14), (3.23); we have

$$\theta = \tan^{-1} \frac{h_2}{h_1} = -f \int \frac{g}{r^2} dv + S(u) \quad (3.26)$$

where  $r^2 = h_1^2 + h_2^2$ ,  $S$  is an arbitrary function of  $u$ . If

$$g[M''(M+N) - M'^2] + [N''(M+N) - N'^2] = 0 \quad (3.27)$$

then  $h_1 = r \cos \theta$ ,  $h_2 = r \sin \theta$  satisfies (3.12)–(3.14). It can be derived, from (3.27), that

$$N'' = \lambda N + \eta_1 \quad (3.28)$$

$$gM'' = -\lambda M + \eta_2 \quad (3.29)$$

where  $\lambda$ ,  $\eta_1$ ,  $\eta_2$  are constants. So  $f$  satisfies

$$f'' = \lambda f \quad (3.30)$$

Conversely, if  $f$  satisfies (3.30), then we can select such  $g, M$  which render (3.27) to hold. Noticing that we only seek the translationally invariant Bäcklund maps here (3.5), we conclude that condition (3.30) is a sufficient condition for the existence of Bäcklund transformations for equation (3.1). The various Bäcklund transformations associated with (3.30) are shown in Table 1.

## References

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