

THE APPLICATION OF COMPATIBLE STRESS ITERATIVE METHOD IN DYNAMIC FINITE ELEMENT ANALYSIS OF HIGH VELOCITY IMPACT*

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Abstract

There is a common difficulty in elastic-plastic impact codes such as EPIC^[2,3], NONSAP^[4], etc.. Most of these codes use the simple linear functions usually taken from static problem to represent the displacement components. In such finite element formulation, the stress components are constant in each element and they are discontinuous in any two neighboring elements. Therefore, the bases of using the virtual work principle in such elements are unreliable. In this paper, we introduce a new method, namely, the compatible stress iterative method, to eliminate the above-said difficulty. The calculated examples show that the calculation using the new method in dynamic finite element analysis of high velocity impact is valid and stable, and the element stiffness can be somewhat reduced.

I. Introduction

There is a common difficulty in elastic-plastic impact codes such as EPIC^[2,3], NONSAP^[4]. Most of these codes use simple linear functions usually taken from static problem to represent the displacement components. In such finite element formulation, the stress components are constant in each element and they are discontinuous in any two neighboring elements. Therefore, the bases of using the virtual work principle in such elements are unreliable.

In this paper, we introduce a new method, namely, the compatible stress iterative method, to eliminate the above-said difficulty. In the new method, we still suppose the displacement components are linear functions in each element but reconstruct the constant stresses into a compatible stress field at each step of calculation. It is obvious that the compatible stress functions and the compatible displacement functions satisfy the needs of the virtual work equation.

With regard to the new method, the mathematical discussions are carried through only with a view to give a reasonable construction of the compatible stress field, but not with all the detailed rigorous proofs a primarily mathematical paper would require, though the validity of the method has been tested empirically on various computational applications. The computational examples show that the calculation using the new method in dynamic finite element analysis of high velocity impact is valid and stable, and the element stiffness can be somewhat reduced.

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II. Compatible Stress Iterative Method

There is a common difficulty in elastic-plastic impact codes previously. Most of these codes use the simple linear functions usually taken from static problem to represent the displacement components. In such finite element formulation, the stress components are constant in each element and they are discontinuous in any two neighboring elements. Therefore, the bases of using the virtual work principle in such elements are unreliable. The virtual work principle which doesn't consider the body force is

$$\int_{\Omega} \rho [\tilde{u}]^T \delta \tilde{u} d\Omega = - \left[\int_{\Omega} [\delta \tilde{\epsilon}]^T \tilde{\sigma} d\Omega - \int_{\Gamma} [\delta \tilde{u}]^T \tilde{q} dT \right] \quad (2.1)$$

where ρ is element density and $\tilde{\sigma}$, \tilde{q} , \tilde{u} are the stress vector, surface force vector, acceleration vector, respectively, and $\delta \tilde{u}$, $\delta \tilde{\epsilon}$ is the virtual displacement vector, virtual strain vector, respectively.

In this paper we introduce a new method namely, the compatible stress iterative method, to eliminate the above-said difficulty. The iterative steps are as follows.

We prefer to use the triangular element in axisymmetrical problems and the tetrahedron element in three-dimensional problems for analysis of high velocity impact due to severe distortion.

First, the element displacement vector, velocity vector, acceleration vector and virtual displacement vector are represented by the linear functions at the time t

$$\tilde{u}^{(e)t} = \sum_{i=1}^k L_i^{(e)t} \tilde{u}_i^{(e)t} \quad (2.2)$$

$$\dot{\tilde{u}}^{(e)t} = \sum_{i=1}^k L_i^{(e)t} \dot{\tilde{u}}_i^{(e)t} \quad (2.3)$$

$$\ddot{\tilde{u}}^{(e)t} = \sum_{i=1}^k L_i^{(e)t} \ddot{\tilde{u}}_i^{(e)t} \quad (2.4)$$

$$\delta \tilde{u}^{(e)t} = \sum_{i=1}^k L_i^{(e)t} \delta \tilde{u}_i^{(e)t} \quad (2.5)$$

where L_i are the area coordinates in axisymmetrical problems or the volume coordinates in three-dimensional problems. k is the number of the element nodes.

Then the constant strain vector or constant strain rate vector of the element can be calculated at the time t

$$\tilde{\epsilon}^{(e)t} = \mathbf{B}^T(\tilde{\nabla}) \tilde{u}^{(e)t} + \tilde{\delta}^{(e)t} \quad (2.6)$$

$$\dot{\tilde{\epsilon}}^{(e)t} = \mathbf{B}^T(\tilde{\nabla}) \dot{\tilde{u}}^{(e)t} + \dot{\tilde{\delta}}^{(e)t} \quad (2.7)$$

where $\mathbf{B}(\tilde{\nabla})$ is the strain operator matrix and $\tilde{\delta}^{(e)t}$ is the strain corrective vector which is caused by the rigid displacement of the element.

In axisymmetrical problems

$$\mathbf{B}(\tilde{\nabla}) = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & \frac{1}{r} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix} \quad (2.8)$$

$$\left. \begin{aligned} \delta_{r_r}^{e,t} &= -\gamma_{rz}^{(e),t} \sin^{-1} \left[\frac{1}{2} \left(\frac{\partial u_x^{(e),t}}{\partial r} - \frac{\partial u_r^{(e),t}}{\partial z} \right) \right] \\ \delta_{r_z}^{e,t} &= \gamma_{rz}^{(e),t} \sin^{-1} \left[\frac{1}{2} \left(\frac{\partial u_x^{(e),t}}{\partial r} - \frac{\partial u_r^{(e),t}}{\partial z} \right) \right] \\ \delta_{\gamma_{rz}}^{e,t} &= 2 \left(\varepsilon_r^{(e),t} - \varepsilon_z^{(e),t} \right) \sin^{-1} \left[\frac{1}{2} \left(\frac{\partial u_x^{(e),t}}{\partial r} - \frac{\partial u_r^{(e),t}}{\partial z} \right) \right] \end{aligned} \right\} \quad (2.9)$$

and in three-dimensional problems

$$\mathbf{B}(\tilde{\nabla}) = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \quad (2.10)$$

$$\left. \begin{aligned} \delta_{\varepsilon_x}^{e,t} &= \gamma_{xz}^{(e),t} \sin^{-1} \left[\frac{1}{2} \left(\frac{\partial u_x^{(e),t}}{\partial z} - \frac{\partial u_z^{(e),t}}{\partial x} \right) \right] \\ &\quad - \gamma_{zy}^{(e),t} \sin^{-1} \left[\frac{1}{2} \left(\frac{\partial u_y^{(e),t}}{\partial x} - \frac{\partial u_x^{(e),t}}{\partial y} \right) \right] \\ \delta_{\gamma_{xy}}^{e,t} &= 2 \left(\varepsilon_x^{(e),t} - \varepsilon_y^{(e),t} \right) \sin^{-1} \left(\frac{1}{2} \left(\frac{\partial u_y^{(e),t}}{\partial x} - \frac{\partial u_x^{(e),t}}{\partial y} \right) \right) \\ \text{etc.} \end{aligned} \right\} \quad (2.11)$$

The strain rate corrective vector $\tilde{\delta}_i^{e,t}$ can be obtained similarly to the $\tilde{\delta}_i^{e,t}$.

According to (2.6) or (2.7), constant stress vector of the element can be calculated at the time t .

Within the elastic limit

$$\tilde{\sigma}_e^{(e),t} = \mathbf{A} \tilde{\varepsilon}^{(e),t} - \tilde{Q}^{(e),t} \quad (2.12)$$

where \mathbf{A} is the matrix of elastic constants and $\tilde{Q}^{(e),t}$ is the artificial viscosity vector which was originally proposed by von Neumann and Richtmyer^[5].

In axisymmetrical problems

$$\mathbf{A} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (2.13)$$

$$[\tilde{Q}^{(e),t}]^T = [\tilde{Q}^{e,t} \quad \tilde{Q}^{e,t} \quad \tilde{Q}^{e,t} \quad 0] \quad (2.14)$$

and in-three dimensional problems

$$A = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (2.15)$$

$$[\tilde{Q}^{(e)t}]^T = [Q^{(e)t} \quad Q^{(e)t} \quad Q^{(e)t} \quad 0 \quad 0 \quad 0] \quad (2.16)$$

The artificial viscosity component is relative to the deformative rate of the element volume and the minimum altitude of the element

$$\left. \begin{aligned} Q^{(e)t} &= C_L (\rho C_s h |\dot{\epsilon}_V|)^{(e)t} + C_0^2 [\rho h^2 (\dot{\epsilon}_V)^2]^{(e)t} & \text{for } \dot{\epsilon}_V^{(e)t} < 0 \\ Q^{(e)t} &= 0 & \text{for } \dot{\epsilon}_V^{(e)t} \geq 0 \end{aligned} \right\} \quad (2.17)$$

Typical values used for the dimensionless coefficients are $C_L = 0.5$ and $C_0^2 = 0.4^{[6]}$.

In the plastic range

$$\tilde{\sigma}_e^{(e)t} = \tilde{S}_e^{(e)t} - (\tilde{P}^{(e)t} + \tilde{Q}^{(e)t}) \quad (2.18)$$

where $\tilde{S}_e^{(e)t}$ is the plastic deviator stress vector of the element, which is determined by von Mises incremental theory. $\tilde{P}^{(e)t}$ is the hydrostatic pressure vector. Its zero components are the same as $\tilde{Q}^{(e)t}$ zero components and its nonzero components are determined by Mie-Grüneisen equation of state.

$$\left. \begin{aligned} S_i^{(e)t} &= \frac{2}{3} \left(\frac{\dot{\epsilon}_i^{(e)t}}{\bar{\epsilon}^{(e)t}} \right) \sigma_i^{(e)t} \\ \text{etc.} \end{aligned} \right\} \quad (2.19)$$

where $\sigma_i^{(e)t}$ is the tensile strength of the material and $\bar{\epsilon}^{(e)t}$ is the equivalent strain rate.

$$P^{(e)t} = (k_1 \mu + k_2 \mu^2 + k_3 \mu^3)^{(e)t} \left(1 - \frac{\Gamma_m \mu}{2} \right)^{(e)t} + \Gamma_m \rho^0 E_m^{(e)t}$$

where $\mu = \rho/\rho_0 - 1 = V^0/V - 1$.

The specific internal energy, $E_m^{(e)t}$ is obtained from the work done on the element by the stress vector. k_1, k_2, k_3 are material-dependent constants and Γ_m is the Grüneisen coefficient.

According to (2.12) or (2.18), the constant stress vector of each element can be obtained but is discontinuous in any two neighboring elements. In order to suit the stress continuous condition of equation (2.1), we reconstruct the constant into a compatible stress field.

$$\tilde{\sigma}_N^{(e)t} = f(\tilde{\sigma}_e^{(e)t}) \quad (2.21)$$

When equations (2.4), (2.5), (2.21) are substituted into equation (2.1) and all elements are fitted together, the node acceleration vector can be determined.

$$\ddot{u}_i^{(e)t} = \frac{\sum_i \tilde{F}_i^{(e)t}}{\sum_i M_i^{(e)t}} \quad (2.22)$$

where $M_i^{(e)t}$ is the lumped mass and $\tilde{F}_i^{(e)t}$ is the force vector at the node i of the element e .

Thus the displacement vector and the coordinates of the node i of the element at the time $t + \Delta t$ can be determined by integrating.

$$\ddot{u}_i^{(e)t+\Delta t} = \ddot{u}_i^{(e)t} + \ddot{u}_i^{(e)t}(\Delta t) \quad (2.23)$$

$$\dot{u}_i^{(e)t+\Delta t} = \dot{u}_i^{(e)t} + \ddot{u}_i^{(e)t}(\Delta t)^{t+} \quad (2.24)$$

$$\bar{x}_i^{(e)t+\Delta t} = \bar{x}_i^{(e)t_0} + \dot{u}_i^{(e)t+\Delta t} \quad (2.25)$$

III. Structure of Compatible Stress and Calculation of Node Force

Since the constant stress vector of each element has been obtained from equation (2.12) or (2.18), we define the stresses of each node by using the weighted average of the nearby element stresses of the node.

$$\tilde{\sigma}_i^{(e)t} = \frac{\sum_i \tilde{\sigma}_e^{(e)t} V^{(e)t}}{\sum_i V^{(e)t}} \quad (3.1)$$

Then the stress vector of each element can be reconstructed, namely, equation (2.21) can be written definitely

$$\tilde{\sigma}_N^{(e)t} = \sum_{i=1}^k N_i^{(e)t} \tilde{\sigma}_i^{(e)t} \quad (3.2)$$

where $N_i^{(e)t}$ is a quadratic form function

$$N_i^{(e)t} = L_i^{(e)t} + \lambda_o L_i^{(e)t} \sum_{i=1}^k L_i^{(e)t} \quad (3.3)$$

The quadratic function $N_i^{(e)t}$ has the following qualities

(1) $N_i^{(e)t} = 1$, $N_j^{(e)t} = 0$ at node i

(2) $N_i^{(e)t} = 0$ at $\Gamma_{\Phi i}^{(e)t}$

where $\Gamma_{\Phi i}^{(e)t}$ is a boundary of the element which doesn't contain the node i .

(3) The coefficient λ_o can be determined by an additional condition. The additional condition we present in this paper is $\tilde{\sigma}_N^{(e)t} = \sigma_e^{(e)t}$, at the center of each element. Therefore,

In axisymmetrical problems

$$\lambda_o = \frac{3}{2} \left(\frac{3\tilde{\sigma}_e^{(e)t}}{\sum_{i=1}^k \tilde{\sigma}_i^{(e)t}} - 1 \right) \quad (3.4)$$

and in three-dimensional problems

$$\lambda_e = \frac{4}{3} \left(\frac{4\bar{\sigma}_e^{(e)t}}{\sum_{i=1}^4 \bar{\sigma}_i^{(e)t}} - 1 \right) \quad (3.5)$$

It is obvious that the stress field which is determined by equation (3.2) is compatible in any two neighboring elements and the following equation holds approximately.

$$\int_{(e)} \bar{\sigma}_N^{(e)t} d\Omega = \int_{(e)} \bar{\sigma}_e^{(e)t} d\Omega \quad (3.6)$$

Thus the force vector of the node in each element can be obtained by substituting equations (3.2) and (2.5) into equation (2.1).

In axisymmetrical problems

$$[\bar{F}_i^{(e)t}]^T = [f_{r_i}^{(e)t}, f_{z_i}^{(e)t}] \quad (3.7)$$

where

$$\left. \begin{aligned} f_{r_i}^{(e)t} = & - \left[\int_{(e)} \sigma_{r_v}^{(e)t} \frac{\partial}{\partial r} L_i d\Omega + \int_{(e)} \tau_{rz_N}^{(e)t} \frac{\partial}{\partial z} L_i d\Omega \right. \\ & \left. + \frac{1}{r} \int_{(e)} \sigma_{\theta\theta}^{(e)t} L_i d\Omega \right] \\ & \text{etc.} \end{aligned} \right\} \quad (3.8)$$

In three-dimensional problems

$$[\bar{F}_i^{(e)t}]^T = [f_{x_i}^{(e)t}, f_{y_i}^{(e)t}, f_{z_i}^{(e)t}] \quad (3.9)$$

where

$$f_{x_i}^{(e)t} = - \left[\int_{(e)} \sigma_{x_v}^{(e)t} \frac{\partial}{\partial x} L_i d\Omega + \int_{(e)} \tau_{xy}^{(e)t} \frac{\partial}{\partial y} L_i d\Omega + \int_{(e)} \tau_{xz_N}^{(e)t} \frac{\partial}{\partial z} L_i d\Omega \right] \quad (3.10)$$

etc

IV. Computation Examples

As computation examples, Fig. 1 shows the calculated deforming process in which steel projectile strikes the titanium alloy target at 600m/s vertically and table 1 shows the comparison between the compatible stress iterative method and the traditional method.

The computation examples show that the calculation using the compatible stress iterative method in dynamic finite element analysis of high velocity impact is valid and stable, and the element stiffness can be somewhat reduced.

Table 1

Time			t_1	t_2	t_3	t_4	t_5	t_6	Shaped plug time
Penetrating depth	At 600m/s	Traditional method	2.84	5.98	8.98	11.70	14.10	16.30	4.17 μ s
		Compatible method	3.00	6.14	9.23	11.98	14.32	16.48	3.23 μ s
	At 700m/s	Traditional method	5.53	3.31	5.17	7.10	9.06	10.80	3.08 μ s
		Compatible method	3.62	3.45	5.26	7.25	9.17	11.02	2.93 μ s

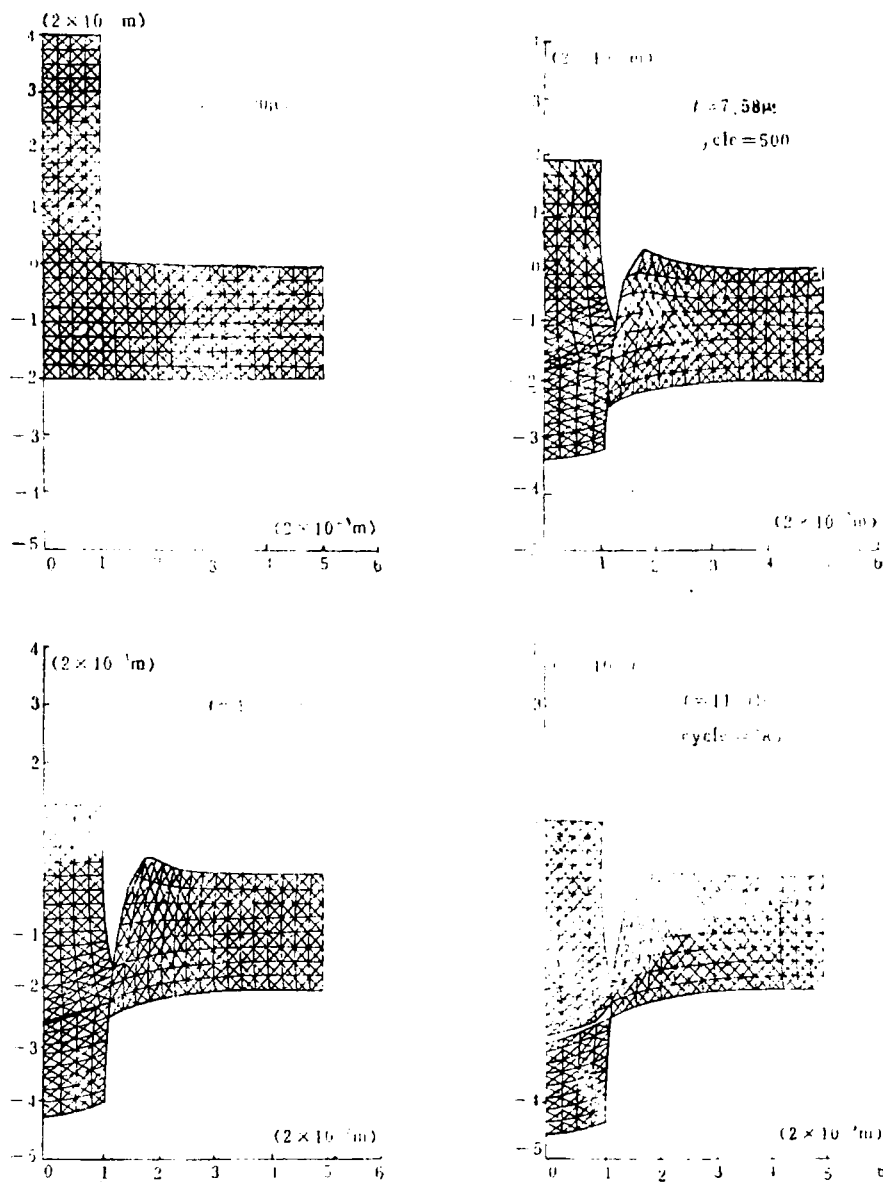


Fig. 1

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