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CONDITIONS FOR INCIPIENT CAVITATION FORMATION*

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Abstract

The growth, equilibrium and stabilization of free gas nucleus are analyzed. It is shown that the cavitation results from growth of free gas nucleus to critical radius and conditions of cavitation have been derived.

I. Crowth of Free Gas Nuclei

A large number of free gas nuclei exist in water. These nuclei are full of undissolved air and vapor. It is the continued growth of these nuclei which forms the cavitation[1].

Consider a free gas nucleus which changes size in a surrounding water. The velocity potential is given by[2]

$$\phi=\frac{R^2}{r}\cdot\frac{dR}{dt} \tag{1.1}$$

in which, $R=R(t)$ is the radius of free gas nucleus, r is the radial distance from its center, t is time.

By formula (1.1), the radial velocity of any water particle relative to the gas nucleus center can be got

$$u=-\frac{\partial\phi}{\partial r}=\frac{R^2}{r^2}\cdot\frac{dR}{dt} \tag{1.2}$$

If the effect of gravity is omitted, the equation of motion for the liquid is[3]

$$-\frac{\partial\phi}{\partial t}+\frac{u^2}{2}+\frac{p}{\rho}=F(t)=\frac{p_\infty}{\rho} \tag{1.3}$$

where ρ is the density of water, p is the pressure at any point of water, p_∞ is the pressure at infinity in the water, where $u=0$, $\phi=0$.

Reducing to $r=R$ gives the motion equation of the free gas nucleus-wall as

$$-R\frac{dU}{dt}-\frac{3}{2}U^2=\frac{p_\infty-p}{\rho} \tag{1.4}$$

Here $U=dR/dt$ is the radial velocity of free gas nucleus-wall, $p=p(R)$ is the water pressure at the outside of free gas nucleus-wall.

A free gas nucleus grows very rapidly. The changing process of its volume can be regarded as an adiabatic process. Hence

$$p=p_1\left(\frac{R_0}{R}\right)^{3\gamma}+p_v-\frac{2\sigma}{R} \tag{1.5}$$

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in which R_0 is the initial radius of nucleus, p_1 is the air pressure inside the nucleus at initial moment $(R=R_0,\ \dot{R}=0)$, $p_v=p_v(T)$ is the vapor pressure inside the nucleus, $\sigma=\sigma(T)$ is the surface tension of water, γ is the gas constant (adiabatic) of air.

Substituting (1.5) into (1.4) gives

$$R\frac{dU}{dt}+\frac{3}{2}U^2=\frac{1}{2R^2U}\cdot\frac{d}{dt}(R^3U^2)=\frac{p-p_\infty}{\rho}$$

$$=C_0^2\left(\frac{R_0}{R}\right)^{3\gamma}-\frac{2\sigma}{\rho R}-\frac{1}{\rho}(p_\infty-p_v) \qquad (1.6)$$

where $C_0^2=p_1/\rho$

Integrating (1.6), we have

$$U=\frac{dR}{dt}=\left\{\frac{2C_0^2}{3(1-\gamma)}\left[\left(\frac{R_0}{R}\right)^{3\gamma}-\left(\frac{R_0}{R}\right)^3\right]+\frac{2\sigma}{\rho R}\left[\left(\frac{R_0}{R}\right)^2-1\right]\right.$$
$$\left.+\frac{2}{3}\frac{(p_\infty-p_v)}{\rho}\left[\left(\frac{R_0}{R}\right)^3-1\right]\right\}^{1/2} \qquad (1.7)$$

or

$$dt=\frac{dR}{\left\{\frac{2C_0^2}{3(1-\gamma)}\left[\left(\frac{R_0}{R}\right)^{3\gamma}-\left(\frac{R_0}{R}\right)^3\right]+\frac{2\sigma}{\rho R}\left[\left(\frac{R_0}{R}\right)^2-1\right]+\frac{2}{3}\frac{(p_\infty-p_v)}{\rho}\left[\left(\frac{R_0}{R}\right)^3-1\right]\right\}^{1/2}} \qquad (1.8)$$

By doing numerical intergrations of formula (1.8), we can find the relationship of time versus nucleus radius under different p_∞ (see Fig. 1 or Table 1).

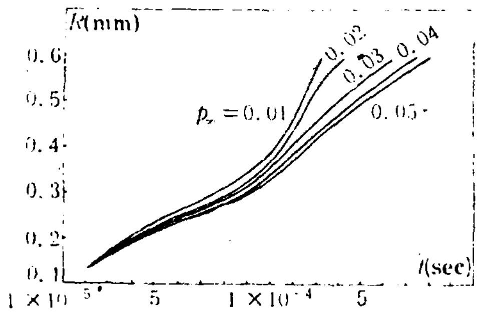

Fig. 1 $R=f(t)$ curve

II. Equilibrium and Stabilization of Free Gas Nucleus

By formula (1.6), we have

$$R\frac{dU}{dt}+\frac{3}{2}U^2=C_0^2\left(\frac{R_0}{R}\right)^{3\gamma}-\frac{2\sigma}{\rho R}-\frac{1}{\rho}(p_\infty-p_v)=f(R,\ T) \qquad (2.1)$$

The function $f(R,T)$ is the force encouraging radius change. The sign of $f(R,T)$ will be positive to promote nucleus growth and negative to encourage collapse.

Let $\partial f/\partial R=0$, for constant temperature, the nucleus radius R_c corresponding to the minimpin value of $f(R,T)$ is

$$R_c=\left(\frac{2\sigma}{3\rho C_0^2R_0^{3\gamma}\gamma}\right)^{\frac{1}{1-3\gamma}} \qquad (2.2)$$

If water temperature T and the initial radius R_0 of a free gas nucleus are given, R_c is a constant value. If $T=15°C$, $R_0=10^{-1}$mm, then $R_c=0.5778$mm.

When $R>R_c$, then $\partial f/\partial R>0$, the nucleus is in an unstable state.

When $R<R_c$, then $\partial f/\partial R<0$, the nucleus is in a stable state.

It is thus clear that under fixed temperature, value of R_c is the maximum radius keeping the nucleus in stability. We call it the critical radius of free gas nucleus. In negative pressure fields, the nucleus will grow unceasingly, its radius will expand gradually, and it is in a stable state until its radius is equal to or exceeds the critical value R_c. At this time, plenty of water around the nucleus is gasified, the nucleus grows very quickly to form a microscopic bubble. This phenomenon is named cavitation.

Supposing the time that a nucleus grows from the initial radius R_0 to R_c is t_{Rc}, the relation between p_∞ and t_{Rc} can be found out from Fig. 1, as is shown in Fig. 2.

Table 1 The relationship of time versus nucleus radius ($T=15$℃, $R_0=10^{-1}$mm)

	$p_\infty=0.01$ (kg/cm²)	$p_\infty=0.02$ (kg/cm²)	$p_\infty=0.03$ (kg/cm²)	$p_\infty=0.04$ (kg/cm²)	$p_\infty=0.05$ (kg/cm²)	$p_\infty=0.1$ (kg/cm²)	$p_\infty=0.2$ (kg/cm²)	$p_\infty=0.3$ (kg/cm²)
R (mm)	t(s)	t(s)	t(s)	t(s)	t(s)	t(s)	t(s)	t(s)
0.12	1.7136×10^{-5}	1.7228×10^{-5}	1.7321×10^{-5}	1.7416×10^{-5}	1.7513×10^{-5}	1.8022×10^{-5}	1.9192×10^{-5}	2.0632×10^{-5}
0.14	2.1627×10^{-5}	2.1760×10^{-5}	2.1695×10^{-5}	2.2032×10^{-5}	2.2173×10^{-5}	2.2922×10^{-5}	2.4714×10^{-5}	2.7107×10^{-5}
0.16	2.6269×10^{-5}	2.6460×10^{-5}	2.6656×10^{-5}	2.6857×10^{-5}	2.7064×10^{-5}	2.8190×10^{-5}	3.1112×10^{-5}	3.6057×10^{-5}
0.18	3.1333×10^{-5}	3.1611×10^{-5}	3.1898×10^{-5}	3.2196×10^{-5}	3.2506×10^{-5}	3.4250×10^{-5}	3.9570×10^{-5}	5.7137×10^{-5}
0.20	3.6936×10^{-5}	3.7342×10^{-5}	3.7766×10^{-5}	3.8211×10^{-5}	3.8678×10^{-5}	4.1466×10^{-5}	5.4376×10^{-5}	6.8412×10^{-5}
0.30	7.4967×10^{-5}	7.7303×10^{-5}	8.0022×10^{-5}	8.3273×10^{-5}	8.7314×10^{-5}	1.5365×10^{-4}	1.3102×10^{-4}	1.0033×10^{-4}
0.40	1.3217×10^{-4}	1.4196×10^{-4}	1.5655×10^{-4}	1.8570×10^{-4}	2.8347×10^{-4}	2.1784×10^{-4}	1.6450×10^{-4}	1.2589×10^{-4}
0.50	2.0952×10^{-4}	2.4160×10^{-4}	3.5985×10^{-4}	4.2563×10^{-4}	3.9404×10^{-4}	2.6720×10^{-4}	1.9526×10^{-4}	1.6011×10^{-4}
0.60	3.0570 10^{-4}	3.9639×10^{-4}	6.6291×10^{-4}	5.3436×10^{-4}	4.7566×10^{-4}	3.1307×10^{-4}	2.2507×10^{-4}	1.7385×10^{-4}

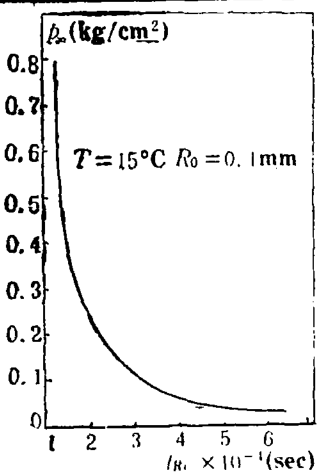

Fig. 2 $t_{Rc}\sim p_\infty$ carve

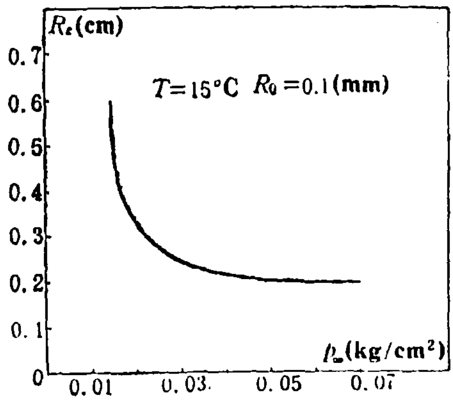

Fig. 3 $R_c\sim p_\infty$ carve

From Fig. 2, we can see, under general negative pressure, the time when the nucleus grows from the initial radius $R_0=10^{-1}$mm to the critical radius R_c is approximately 10^{-4} second. In other words, even if the pressure fields are of high frequency oscillation, the time is long enough to make the nucleus grow to the critical radius.

If $f(R,T)=0$, the nucleus is in the equilibrium state, its radius is correspondingly named the equilibrium radius R_e. From (2.1), the following equation can be obtained that the equilibrium radius should satisfy

$$C_0^2\left(\frac{R_0}{R_e}\right)^{3\gamma}-\frac{2\sigma}{\rho R_e}-\frac{1}{\rho}(p_\infty-p_v)=0 \tag{2.3}$$

When the water temperature and R_0 are given, the value of R_e will vary with p_∞ their relationship is illustrated in Fig. 3.

From Fig. 3 we see that a free gas nucleus has a definite equilibrium radius when p_∞ is in certain range, within which the nucleus will grow till $R=R_e$ and will not form cavitation. p_∞ has a limit value at the vicinity of 0.015 kg/cm^2, where the curve slope tends towards infinity, which indicates that when p_∞ is smaller than this limit value, a nucleus has no equilibrium state and will grow continually until $R=R_c$, thus forming cavitation. This limit value of p_∞ can be defined as follows.

According to the preceding discussion, when $R=R_c$, function $f(R,T)$ has minimum value. By (2.3), we have

$$f(R,\ T)_{\min}=\left[C_0^2\left(\frac{R_0}{R}\right)^{3\gamma}-\frac{2\sigma}{\rho R}-\frac{1}{\rho}(p_\infty-p_v)\right]_{\min}$$

$$=\left[C_0^2\left(\frac{R_0}{R_c}\right)^{3\gamma}-\frac{2\sigma}{\rho R_c}-\frac{1}{\rho}(p_\infty-p_v)\right]=0 \qquad (2.4)$$

By which the minimum value of p_∞ that the nucleus is in the equilibrium state is obtained.

$$p_{\infty\min}=\rho C_0^2\left(\frac{R_0}{R_c}\right)^{3\gamma}-\frac{2\sigma}{R_c}+p_v \qquad (2.5)$$

By (2.2) we have the solution of $\rho C_0^2 R_0^{3\gamma}$, then substituting (2.5) with it, we get

$$p_{\infty\min}=\frac{2\sigma(1-3\gamma)}{3R_c\gamma}+p_v \qquad (2.6)$$

When water temperature $T=15°\mathrm{C}$, $R_0=10^{-1}$mm, by (2.6) we get $p_{\infty\min}=0.01542\mathrm{kg/cm^2}$

III. Conditions for Incipient Cavitation Formation

According to the preceding discussion, the conditions of incipient cavitation formation can be obtained as follows:

1. It does not form cavitation, if $p_\infty>p_{\infty\min}$
2. It forms cavitation, if $p_\infty\leqslant p_{\infty\min}$.

For a submerged body, p_∞ is the minimum pressure of the body surface. For the oscillating fields, the oscillation of pressure can be omitted, because half of the cycle is much greater than t_{Rc} and $p_\infty=p_a-p$, where p_a is the hydrostatic pressure at the oscillator, p is the crest value of the pressure produced by the oscillator.

IV. Additional Remarks

In this paper the conditions for incipient cavitation formation are discussed in water. These conditions may be applicable to any other liquids. The compressibility of water is omitted in this discussion, because only at the last stage of the bubble collapse does it show the remarkable effect[3].

The effect of viscosity is to produce damping and loss of mechanical energy during the growth and collapse process. The viscosity of water is very low, so that the effects of viscosity on cavitation are relatively negligible[3].

References

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