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ON THE TWO BIFURCATIONS OF A WHITE-NOISE EXCITED HOPF BIFURCATION SYSTEM

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Abstract

The present work is concerned with the behavior of the second bifurcation of a Hopf bifurcation system excited by white-noise. It is found that the intervention of noises induces a drift of the bifurcation point along with the subtantial change in bifurcation type.

Key words white-noise, parametric excitation, stochastic averaging method

I. Introduction

Nonlinear analysis efforts mainly include researches on the stable motion of a system. investigations on its stability features and the instantaneous motion of a dynamical system when changes occur to its governing parameters. The so-called stochastic bifurcation implies the transition phenomenon that arises, under the action of noises, in a nonlinear system in the vicinity (neighbourhood) of bifurcation point.

Prigoging and Nicolis[1], Haken[2], and Graham[3] noticed the significant effect of noises on the long-time behavior of a system far from equilibrium state and that on the non-equilibrium phase transition, and by the end of 1970s, these investigators initiated investigations on stochastic bifurcation behaviors.

With the intervention of noises, a complete description of a nonlinear stochastic system usually implies:

(1) for white-noise systems, Ito stochastic differential equation is satisfied by the sample functions of state variables;

(2) for an Ito system, the probability distribution function of sample orbits satisfies FPK equation.

This is attributable to the emergence of the two types of conceptual guidelines in stochastic bifurcation studies.

In the early stages, research interests were mainly focused on the mathematics and physics essence of stochasatic bifurcations, starting with the invariant measure——the stationary solution of FPK equation of a white-noise system, in order to determine the location of the bifurcation point and the form of the bifurcation solutions.

As the process of evolution keeps going forward, it is increasingly more realized that stochastic bifurcation is essentially a kind of nonlinear singular phenomenon that, appearing in the sample orbits of the stochastic system, reflects the catastrophe mechanism of the sample

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stability of a system. As to the invariant measure method, its capability of grasping the information on stochastic bifurcations is less completed[4, 6, 22]. As revealed by the ergodic theory, the invariant measure $\mu(x)$ is, in essence, a measure of the averaging time of an arbitrary orbit in the infinitesimal neighborhood of x. Therefore, it is incapable of exactly describing the state of the sample orbit. Still, only in certain extent, does the extremum of the invariant measures reflects the most probable and almost improbable motion of a system.

On the basis of the advances achieved in investigations on sample stability problems regarding nonlinear stochastic systems[7, 8], significant amount of effort has been devoted to the research topics in connection with sample stability problems of nonlinear stochastic systems ever since the advent of 1980. In specific, such effort has been focused on the evolution of maximum Lyapunov exponent[9, 10, 11, 12] of relevant systems. In this aspect, the maximum Lyapunov exponent is employed as an important index for the definition of stochastic bifurcation point in probability 1 sense. At the same time, specific attention was attached to investigations on the geometrical attributes[13] of sample orbits of the diffusion solution processes to nonlinear stochastic dynamical systems. Apparently, the core of investigations is consistent with the conceptual guidelines manifested by the increasingly wide-spread ergodic theory in research activities regarding the theory of deterministic dynamical system.

Based on the classical Khasminskii method, the maximum Lyapunov exponent of a linearized stochastic system at its equilibrium point is evaluated, and, in addition, the location of the first bifurcation point in probability 1 sense is determined. Next, the stochastic averaging method is invoked for investigation on the invariant measure of FPK equations, corresponding to the amplitude Ito stochastic differential equations. Also involved in these investigations are the determination of the second bifurcation point and bifurcation solution in maximum probablility sense. It is found that, the intervention of noises does essentially induce a change of the original bifurcation type.

II. Model of the Hopf Bifurcation System Parametrically Excited by White-Noise

Consider a model of a kind of typical Hopf system which is excited parametrically by stochastic perturbations

$$\begin{aligned} \dot{u} &= \mu u + \omega_0 v - \varepsilon u^3 + u\sigma_1\xi_1(t) \\ \dot{v} &= -\omega_0 u + \mu v - \varepsilon u^2 v + v\sigma_2\xi_2(t) \end{aligned} \tag{2.1}$$

where μ is the bifurcation parameter, ω_0 a constant, and ε is a small quantity. Again $\xi_1(t)$ and $\xi_2(t)$ represent independent unit white-noise processes respectively. Note also that stochastic differential equation (2.1) is established in Stratonovitch sense. In addition, σ_1 and σ_2 represent respectively the intensity of $\xi_1(t)$ and $\xi_2(t)$, σ_1^2, σ_2^2 and μ being small quantities of the same order as ε.

For a deterministic Hopf Bifurcation system($\sigma_1=\sigma_2=0$),owing to the deterioration of the real part of the eigenvalue of the Lyapunov matrix for its linearized system, a limit circle emerges at the equilibrium point ($x=x=0$). To examine the effect of noise on the stability and bifurcation behavior of a Hopf bifurcation system, the maximum Lyapunov exponent and rotation number of the linearized system

$$\begin{aligned} \dot{u} &= \mu u + \omega_0 v + u\sigma_1\xi_1(t) \\ \dot{v} &= -\omega_0 u + \mu v + v\sigma_2\xi_2(t) \end{aligned} \tag{2.2}$$

corresponding to equation (2.1) will be determined.

Taking into account Wong-Zakai correction terms[14] gives rise to Ito stochastic differential equation

$$du = Audt + B_1 u dW_1 + B_2 u dW_2 \tag{2.3}$$

$$A = \begin{pmatrix} \mu + \frac{\sigma_1^2}{2} & \omega_0 \\ -\omega_0 & \mu + \frac{\sigma_2^2}{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad u = \begin{pmatrix} u \\ v \end{pmatrix}$$

where $W_1(t)$, $W_2(t)$ are independent Wiener processes. Also, corresponding to (2.3), the vector solution process u is a diffusion process[14].

III. Khasminskii Transform and the Invariant Measure of One-Dimension Diffusion Process

By means of Khasminskii transform[8], the diffusion process u is mapped on a unit circle. Using transform

$$s_1 = \cos\theta = \frac{u}{\|u\|}, \quad s_2 = \sin\theta = \frac{v}{\|u\|} \tag{3.1}$$

denoting

$$s = (s_1, s_2)^T = (\cos\theta, \ \sin\theta)^T, \quad \rho = \ln\|u\|, \quad \|u\| = (u^2+v^2)^{\frac{1}{2}} \tag{3.2}$$

and applying Ito differentiation rules[14] to the manipulations with respect to ρ and $\theta\,(=\mathrm{arctg}(v/u))$ yields the Ito stochastic differential equation

$$d\rho = Q(\theta)dt + \Sigma_r(\theta)dW_r(t) \tag{3.3}$$

$$d\theta = \Phi(\theta)dt + \Psi(\theta)dW_\theta(t) \tag{3.4}$$

in terms of ρ and θ, on the basis of equation (3.2). Note that in the above equation $W_r(t)\,(r=1,2)$, $W_\theta(t)$ are independent Wiener processes. Also,

$$Q(\theta) = s^T As + \frac{1}{2}\mathrm{tr}B - s^T Bs \tag{3.5}$$

$$\Sigma_r(\theta) = s^T B_r s \qquad (r=1,2) \tag{3.6}$$

$$\Phi(\theta) = -\tilde{s}^T As + \tilde{s}^T Bs \tag{3.7}$$

$$\Psi^{-2}(\theta) = \tilde{s}^T Bs \tag{3.8}$$

In this process, setting

$$B(\theta) = \sum_{r=1}^{2} (B_r s)(B_r s)^T \tag{3.9}$$

$$\tilde{s} = -\frac{ds}{d\theta} = (s_2, -s_1)^T = (\sin\theta, -\cos\theta)^T$$

yields, corresponding to (3.5)~(3.8), the following equations

$$Q(\theta) = \mu + (\sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta) - (\sigma_1^2\cos^4\theta + \sigma_2^2\sin^4\theta) \tag{3.10}$$

$$\Sigma_1(\theta)=\sigma_1\cos^2\theta,\quad \Sigma_2(\theta)=\sigma_2\sin^2\theta \tag{3.11}$$

$$\Phi(\theta)=-\omega_0+\frac{1}{2}(\sigma_2^2-\sigma_1^2)\sin\theta\cos\theta+(\sigma_1^2\cos^2\theta-\sigma_2^2\sin^2\theta)\sin\theta\cos\theta \tag{3.12}$$

$$\Psi^2(\theta)=(\sigma_1^2+\sigma_2^2)\sin^2\theta\cos^2\theta \tag{3.13}$$

From Ref. [8], the differential generator relevant to (2.3) is

$$\begin{aligned}L&=\Phi(\theta)\frac{d}{d\theta}+\frac{1}{2}\Psi^2(\theta)\frac{d^2}{d\theta^2}\\&=\left[-\omega_0+\frac{1}{2}(\sigma_2^2-\sigma_1^2)\sin\theta\cos\theta+(\sigma_1^2\cos^2\theta-\sigma_2^2\sin^2\theta)\sin\theta\cos\theta\right]\frac{d}{d\theta}\\&\quad+\left[\frac{1}{2}(\sigma_1^2+\sigma_2^2)\sin^2\theta\cos^2\theta\right]\frac{d^2}{d\theta^2}\end{aligned} \tag{3.14}$$

Noticing the form of equation (3.4) together with its coefficients expressions (3.12), (3.13), it is apparent that θ is an one-dimension diffusion process on unit circle, whose stationary probability density function——invariant measure $\mu(\theta)$ satisfies the following FPK equation[14]

$$L^*\mu(\theta)=0 \tag{3.15}$$

with L^* as the adjoint operator corresponding to L, i. e.

$$\frac{1}{2}\frac{d^2}{d\theta^2}[\Psi^2(\theta)\mu(\theta)]-\frac{d}{d\theta}[\Phi(\theta)\mu(\theta)]=0 \tag{3.16}$$

Via direct integration of (3.15), the general solution of $\mu(\theta)$ is obtained as

$$\mu(\theta)=\frac{C}{\Psi^2(\theta)W(\theta)}+\frac{G}{\Psi^2(\theta)W(\theta)}\int W(\theta)d\theta \tag{3.17}$$

where

$$W(\theta)=\exp\left[-2\int\Phi(\theta)\Psi^{-2}(\theta)d\theta\right] \tag{3.18}$$

with C, G as integration constants to be determined.

The decisive effect of the singular points of the diffusion process θ, located on a unit circle, on the concrete form of $\mu(\theta)$——the invariant measure, is revealed by the classical theory of one dimensional diffusion processes[15]. Let us examine the characteristics of singularities of θ on a unit circle.

It is apparent that for $\theta=0,\ \pi/2,\ \pi,\ 3\pi/2$

$$\Phi(\theta)=-\omega_0+\frac{1}{2}(\sigma_2^2-\sigma_1^2)\sin\theta\cos\theta+(\sigma_1^2\cos^2\theta-\sigma_2^2\sin^2\theta)\sin\theta\cos\theta=-\omega_0<0$$

$$\Psi^2(\theta)=(\sigma_1^2+\sigma_2^2)\sin^2\theta\cos^2\theta=0$$

From the definition of singularities[16], it is known that 0, $\pi/2$, π, $3\pi/2$, are the left shunt singularities of the diffusion process θ, while the others on the circle are non-singular points. Additionally, in accordance with Kozin and Prodromou[17], R. Mitchell and Kozin[18], and Nishioka[19], one, naturally, relevant to the concrete form of $\mu(\theta)$, comes down to the conclusions:

(1) On the unit circle, forms of invariant measure at points, symmetrical with respect to

the center of the circle, are identical i. e.

$$\mu(\theta)=\mu(\theta\pm\pi) \tag{3.19}$$

(2) In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of θ, invariant measures are

$$\mu(\theta)=\begin{cases} GF(\theta) & \left(-\frac{\pi}{2}<\theta<0\right) \\ GF\left(\theta-\frac{\pi}{2}\right) & \left(0<\theta<\frac{\pi}{2}\right) \end{cases} \tag{3.20}$$

where

$$F(\theta)=\Psi^{-2}(\theta)W^{-1}(\theta)\int_{-\pi/2}^{\theta} W(\varphi)d\varphi \tag{3.21}$$

Calculations on (3.18), (3.20) and (3.21) lead to the following form

$$\mu(\theta)=\frac{G}{\sigma_1^2+\sigma_2^2}\exp(\gamma f(\theta))(\sin\theta)^{\alpha-2}(\cos\theta)^{\beta-2}\int_{-\pi/2}^{\theta}\exp[-\gamma f(\varphi)]\sin^{-\alpha}\varphi\cos^{-\beta}\varphi d\varphi \tag{3.22}$$

as $\theta\in[-\pi/2, 0]$. Note in the above equation (3.22)

$$f(\theta)=\mathrm{ctg}\theta-\mathrm{tg}\theta \tag{3.23}$$

$$\gamma=\frac{2\omega_0}{\sigma_1^2+\sigma_2^2}, \quad \alpha=\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}, \quad \beta=\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \tag{3.24}$$

In addition, integration constant G is available via the normalization condition of $\mu(\theta)$

$$\int_0^{2\pi}\mu(\theta)d\theta=1$$

such that

$$G^{-1}=2\left(\int_{-\frac{\pi}{2}}^{0}F(\theta)d\theta+\int_0^{\frac{\pi}{2}}F\left(\theta-\frac{\pi}{2}\right)d\theta\right) \tag{3.25}$$

IV. Maximum Lyapunov Exponent and Rotation Number

It is known from [16] that, on the entire unit circle, the diffusion process θ is ergodic, as long as the singularities of θ on the unit circle are left shunt points. On such basis, from the Oseledec Multiplicative Ergodic Theorem[13] arises the maximum Lyapunov exponent λ and rotation number α, relative to the stochastic differential system (2.3), i. e.

$$\lambda=\lim_{t\to+\infty}\frac{1}{t}\ln\,(u^2+v^2)^{\frac{1}{2}} \tag{4.1}$$

$$\alpha=\lim_{t\to+\infty}\frac{1}{t}\mathrm{arctg}\left(\frac{v}{u}\right) \tag{4.2}$$

They are determined via the following expressions

$$\lambda=E[Q(\theta)]=\int_0^{2\pi} Q(\theta)\mu(\theta)d\theta \tag{4.3}$$

$$\alpha=E[\Phi(\theta)]=\int_0^{2\pi} \Phi(\theta)\mu(\theta)d\theta \tag{4.4}$$

The Lyapunov exponent reflects the average exponent change rate of system (2.1), while the rotation number reflects the average rotation rate of unit vector (s_1, s_2).

Let the maximum Lyapunov exponent be considered first. Substituting (3.10), (3.22)~(3.24) into (4.3), and using integral transfom $u=\mathrm{tg}\theta$, yields

$$\lambda=\frac{\int_{-\infty}^{+\infty} p(u)\gamma_1\exp[\gamma f(u)]u^{-1}du\int_{-\infty}^{u}\exp[-\gamma f(v)]v^{-1}dv}{\int_{-\infty}^{+\infty}\gamma_1\exp[\gamma f(u)]u^{-1}du\int_{-\infty}^{u}\exp[-\gamma f(v)]v^{-1}dv} \tag{4.5}$$

wherein

$$f(u)=\frac{1}{u}-u \tag{4.6}$$

$$p(u)=\mu+\frac{1}{1+u^2}(\sigma_1^2+\sigma_2^2u^2)-\frac{1}{(1+u^2)^2}(\sigma_1^2+\sigma_2^2u^4) \tag{4.7}$$

$$\gamma_1=\frac{1}{\sigma_1^2+\sigma_2^2} \tag{4.8}$$

Analytic expressions of the exact complete integrations in equation (4.5) are unavailable since the integrals involved are transcendental. The fact that σ_1^2 and σ_2^2 are small quantities of the same order of magnitude as ε and that ω_0 is a positive constant, leads to γ, $\gamma_1\to+\infty$, both being very large. For such cases, the asympototic integration of (4.5) can be evaluated by using the following Laplace asymptotic integration theorem.

Theorem (Laplace)[20]. Let $\varphi(x)$ and $h(x)$ be real continuous functions defined on finite or semi-finite interval $[\alpha, \beta]$ and

(1) for every γ, $\varphi(x)\exp[h(x)]$ be absolutely integrable on $[\alpha, \beta]$;

(2) $h'(x)$ and $h''(x)$ are continuous functions on $[\alpha, \beta]$;

(3) $h(x)$ attain its maxima at α with $h'(x)<0$.

Then, as $\gamma\to+\infty$,

$$\int_\alpha^\beta \varphi(x)\exp[\gamma h(x)]dx\approx-\varphi(\alpha)\exp(\gamma h(\alpha))\frac{1}{h'(\alpha)\gamma}$$

In case, in the above theorem, (3) is replaced by

(3)′ $h(x)$ attains its maxima at β, with $h'(\beta)>0$, then as $\gamma\to+\infty$

$$\int_\alpha^\beta \varphi(x)\exp[\gamma h(x)]dx\approx\varphi(\beta)\exp(\gamma h(\beta))\frac{1}{h'(\beta)\gamma}$$

By using Laplace asymptotic integration theorem, it can be seen that, as $\sigma_1^2+\sigma_2^2\to0$

$$\lambda=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\mu+(\sigma_1^2\cos^2\theta+\sigma_2^2\sin^2\theta)-(\sigma_1^2\cos^4\theta+\sigma_2^2\sin^4\theta)]d\theta$$

$$=\mu+\frac{1}{8}(\sigma_1^2+\sigma_2^2)+o(\sigma_1^2+\sigma_2^2)^2 \qquad (4.9)$$

and

$$\alpha=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{-\omega_0+\left[\frac{1}{2}(\sigma_2^2-\sigma_1^2)+(\sigma_1^2\cos^2\theta+\sigma_2^2\sin^2\theta)\right]\sin\theta\cos\theta\right\}d\theta \qquad (4.10)$$
$$=-\omega_0+o(\sigma_1^2+\sigma_2^2)^2$$

It is evident from (4.9) that, as $\lambda\geqslant 0$ i. e. $\mu\geqslant-(\sigma_1^2+\sigma_2^2)/8$, the sample orbit of system (2.1) becomes unstable in probability 1 sense. Consequently, $\mu=-(\sigma_1^2+\sigma_2^2)/8$ is the bifurcation point of the white-noise perturbed Hopf bifurcation system (2.1). The appearance of this bifurcation point is ahead of time as compared to that of the deterministic Hopf bifurcation point $\mu=0(\sigma_1=\sigma_2=0)$. It is in such sense that the intervention of noise terms weakens the stability of the original system.

V. Invariant Measure, Extremum and the Noise-Induced Second Bifurcation Phenomenon

Corresponding to the stochastic dynamical system (2.1) with time-homogeneous diffusion process solutions, the invariant measure $\mu(u)$ is defined as the solution of the FPK equation of the system, namely, the limit of transition probability density function $p(u,t|u_0)$ as $t\to+\infty$, if such limit exists. In principle, invariant measure is usually correlated with the stationary solution of the stochastic dynamical system (2.1). And, a stationary solution process u of (2.1) implies that (u, ξ) is a stationary vector process on the probability space $(M\times\Omega,\ \mu)$. Ω corresponds to the sample space of Wiener process, M the phase space of the sample orbit $u=u(t,u_0,\omega)$. The momentous significance exhibited by the stationary solution to a stochastic dynamical system is similar to that of the stationary solution (equilibrium point, zero point, fixed point) to a deterministic dynamical system. Whereas, the invariance property of an invariant measure indicates that the measure remains unchanged in the process of the calculation defined on the semi-group of the drift operators which are defined on Ω. In fact, together with the "invariance" of the sample orbits regarding the calculations in the semi-group of point-map in the phase space M, the above property constitutes jointly the fundamental premise to the definition of stochastic flow and stochastic dynamical system on the probability space $(\Omega\times M,\ \mu)$. In essence, the existence of the invariant measure is the sufficient-necessary condition to the stationarity of (u, ξ) and vice versa.

The physical significance of the invariant measure is ascribed to its capability of reflecting the probability distribution of the long-time behaviors of a stochastic dynamical system in state space, i.e. the invariant measure is an important description of the long-time characteristics of the system. It is, and will remain, an important auxiliary characteristic quantity in the study of stochastic bifurcation, even though the information it grasps regarding the bifurcation phenomenon appeared in a nonlinear stochastic system is not complete. In fact, its mechanism to judge the stability situations of a stochastic system is consistent with the relevant method of the classical exit problem. More than this, together with the transition time, it constitutes an effective characteristic quantity[21, 22] for the depiction of the difference between the noise-excited transition phenomena and general chaotic motions.

To achieve further enhancement in depicting the noise-excited bifurcation behaviors, the second bifurcation phenomenon of the stochastic Hopf bifurcation system is examined, which is likely to occur after the first having been experienced. Bifurcation solutions, in most probable sense, will also be investigated.

By introducing new variables $a(t)$ and $\varphi(t)$, and using

$$u=a(t)\sin\phi(t),\quad v=a(t)\cos\phi(t),\quad \phi(t)=\omega_0 t+\varphi(t) \tag{5.1}$$

the following standard equations are obtained

$$\dot{a}=\left[\mu a-\frac{\varepsilon a^3}{2}(1-\cos 2\phi)\right]+\frac{\varepsilon^{\frac{1}{2}}a}{2}[(a_1\xi_1(t)+a_2\xi_2(t))$$

$$+(a_2\xi_2(t)-a_1\xi_1(t))\cos 2\phi] \tag{5.2}$$

$$\dot{\varphi}=\frac{\varepsilon^{\frac{1}{2}}}{2}\sin 2\phi(a_1\xi_1(t)-a_2\xi_2(t)) \tag{5.3}$$

where

$$\varepsilon^{\frac{1}{2}}a_1=\sigma_1,\quad \varepsilon^{\frac{1}{2}}a_2=\sigma_2 \tag{5.4}$$

and a_1, a_2 are constants. Note that in the above equation, $a(t)$ and $\varphi(t)$ represent amplitude and phase respectively, both being considered as slowly-varied stochastic processes. By virtue of stochastic averaging method, from the above equations arises the amplitude Ito stochastic differential equattion

$$da=m_a dt+\sigma_a dW_a(t) \tag{5.5}$$

where $W_a(t)$ is unit Wiener process, m_a the drift coefficient, and σ_a the diffusion coefficient, and

$$m_a=\left(\mu'+\frac{m_{at}}{2}\right)a-\frac{\varepsilon}{2}a^3 \tag{5.6}$$

$$\sigma_a=m_a^{\frac{1}{2}}a \tag{5.7}$$

$$\mu'=\mu+\frac{\varepsilon}{8}(a_1^2+a_2^2)\pi K \tag{5.8}$$

$$m_{at}=\frac{3\varepsilon}{8}(a_1^2+a_2^2)\pi K \tag{5.9}$$

Note that in the above equations, K is the constant spectral density function of the white-noise process $W_a(t)$. Corresponding to equation (5.5), the FPK equation is

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial a}\left\{\left[\left(\mu'+\frac{m_{at}}{2}\right)a-\frac{1}{2}a^3\right]p\right\}+\frac{1}{2}\frac{\partial^2}{\partial a^2}[m_{at}a^2 p] \tag{5.10}$$

with initial condition

$$p(a,t\,|\,a_0,t_0)\to\delta(a-a_0),\ t\to t_0 \tag{5.11}$$

where $p(a,t\,|\,a_0,\ t_0)$ is the transition probability density function of the amplitude diffusion process $a(t)$ whose invariant measure $\mu(a)$ satisfies the following degenerated FPK equation

$$-\frac{\partial}{\partial a}\left\{\left[\left(\mu'+\frac{m_{at}}{2}\right)a-\frac{1}{2}a^3\right]\mu(a)\right\}+\frac{1}{2}\frac{\partial^2}{\partial a^2}[m_{at}a^2\mu(a)]=0 \tag{5.12}$$

Direct integration of (5.12) yields

$$\mu(a)=\left(\frac{1}{m_{at}}\right)^m\frac{a^{2m-1}}{\Gamma(m)}\exp\left(-\frac{a^2}{2m_{at}}\right) \tag{5.13}$$

where

$$m=\frac{\mu'}{m_{at}}>0,\quad \Gamma(m)=\int_0^{+\infty}\exp[-t]t^{m-1}dt$$

with $\Gamma(m)$ as Gamma function.

Namachchivaya recognized the momentous significance of the extremum of invariant measure $\mu(a)$. As indicated in [5],

(1) As one of the most notable characteristics of μ(a), the number and location of extremum points embody basic information relevant to stationary behaviors of a nonlinear stochastic system.

(2) As an extension of the stationary behavior of the deterministic system, the extremum of $\mu(a)$ tends to represent the stationary behavior of the deterministic system, as the noise intensity approaches zero.

(3) For an ergodic process $a(t)$, as indicated by the Oseledec multiplicative ergodic theorem, $\mu(a)$ is measure of time that the sample orbit stays in the neighbourhood of a. Accordingly, maximum value indicates the longest period of time that the sample orbit stays at the maximum point, implying stability. And, instability, if otherwise.

For the FPK equation (5.12), amplitude $\bar{a}$, in most probable sense, is available from the maximum problem

$$\left.\frac{d\mu(a)}{da}\right|_{a=\bar{a}}=0,\quad \left.\frac{d^2\mu(a)}{da^2}\right|_{a=\bar{a}}<0 \tag{5.14}$$

Based on the first equation of (5.14), one obtains

$$a^{2m-1}>0,\quad a^2=2\mu'-m_{at} \tag{5.15}$$

To satisfy the inequality in (5.15), it is necessarily required that

$$2m-1>0$$

i. e. $\bar{a}=0$, when

$$\mu>\frac{\pi K}{16}(\sigma_1^2+\sigma_2^2) \tag{5.16}$$

It follows from the second equation of (5.14) that

$$\left.\frac{d^2\mu(a)}{da^2}\right|_{a=0}=0$$

when $\mu>\frac{\pi K}{16}(\sigma_1^2+\sigma_2^2)$. And, the second equation of (5.15) yields

$$a^2=2\mu'-m_{at}=2\mu-\frac{\pi K}{8}(\sigma_1^2+\sigma_2^2)$$

From the above equation, one must have $2\mu' - m_{al} \geqslant 0$, that is

$$\mu \geqslant \frac{\pi K}{16}(\sigma_1^2 + \sigma_2^2) \tag{5.17}$$

In addition, from the second equation of (5.14), one arrives at

$$\frac{d^2\mu(a)}{da^2} < 0$$

at $\bar{a} = (2\mu - 2\mu_2)^{\frac{1}{2}}$, when $\mu \geqslant \mu_2\left(=\frac{\pi K}{16}(\sigma_1^2 + \sigma_2^2)\right)$. In other words, the maximum value of $\mu(a)$ is available at $\bar{a}$ when $\mu \geqslant \mu_2$. Whereas $a = (2\mu - 2\mu_2)^{\frac{1}{2}}$ corresponds to the "most coloured" limit circle in the the most probable sense in the phase space. And in phase space, from the long-time behaviors of the sample orbits of the bifurcation solutions to system (2.1), a region with indistinct bound emerges. In fact, this limit circle implies the most probable motion of the long-time characteristics of the sample orbits relevant to the average amplitude $a(t)$. No doubt, μ_2 can be taken as a bifurcation point of the system in the most probable sense, except for the bifurcation point $\mu = \mu_1\left(= -\frac{1}{8}(\sigma_1^2 + \sigma_2^2)\right)$ in probability 1 sense.

In consequence of the above discussion, it is evident that there are two bifurcation points in system (2.1)—bifurcation point μ_1 in probability 1 sense, and bifurcation point $\mu_2 (\mu_1 < 0 < \mu_2)$ in the most probable sense. As long as μ_1 shows up ahead of the bifurcation point $\mu = 0$ of the deterministic Hopf bifurcation system, the appearance of μ_1, in effect, renders the instability of the system shifted earlier. As μ passes μ_1 while moving along μ-axis in positive direction, the trivial solution $a = 0$ loses stability (in probability 1 sense), bifurcation sets in and nontrivial $a(t)$ emerges. However, at this moment, the nontrivial $a(t)$ is still not capable of representing a limit circle. The concrete form of $a(t)$ is yet to be determined. It is appropriate, therefore, not to have μ_1 considered as Hopf bifurcation point in probability 1 sense. As μ passes μ_2, limit circle in the most probable sense appears. However, this limit circle still can not be considered as being bifurcated from $a = 0$, as far as $a = 0$ is in almost sure sense unstable when $\mu_1 \leqslant \mu \leqslant \mu_2$. Meanwhile, μ_2 can not be considered as Hopf bifurcation point either.

VI. Concluding Remarks

Investigations on the maximum Lyapunov exponent and rotation number along with the invariant measure attribute of a Hopf bifurcation system excited parametrically by white-noise are conducted. It is found that, in the present effort, the type of the Hopf bifurcation system is thoroughly changed when parametrically excited by white-noise, along with the occurrence of a changing in the location of the bifurcation point. Limit circle appears to the system, in the most probable sense, after the bifurcation parameter passes the second bifurcation point. Whereas, limit circles do not appear after μ passes the first bifurcation point. Concrete form of the relevant bifurcation solutions is yet to be investigated.

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