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A NOTE ON PROBABILISTIC INTERPRETATION FOR QUASILINEAR MIXED BOUNDARY PROBLEMS

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Abstract

Solutions of quasilinear mixed boundary problems for the some parabolic and elliptic partial differential equations are interpreted as solutions of a kind of backward stochastic differential equations, which are associated with the classical Ito forward stochastic differential equations with reflecting boundary conditions.

Key words quasilinear partial differential equations, mixed boundary problems, stochastic differential equations, probabilistic interpretations

I. Introduction

We know that the probabilistic representation of solutions of partial differential equations with the mixed boundary conditions has many important applications both in theory of partial differential equations and in that of stochastic differential equations with reflecting boundary conditions, such as optimal control theory, submartingale problem, variational and quasivariational inequality, etc (see [1], [2] and [4] etc).

Let $a(x)=\{a_{ij}(x)\}_{i,j=1}^{d}$ and $b(x)=\{b_i(x)\}_{i=1}^{d}$ be the bounded Borel measurable matrixvalued functions, and let L_x be an operator defined by:

$$L_x u(x)=\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^{2}u(x)+\sum_{i=1}^{d} b_i(x)\partial_i u(x) \tag{1.1}$$

where $\partial_{ij}u(x)=\frac{\partial^2}{\partial x^i\partial x^j}u(x)$ and $\partial_i u(x)=\frac{\partial}{\partial x^i}u(x)$ for all $j=1, 2, \ldots, d$.

Let D be a bounded open set in R^d with the smooth boundary $\partial D=\Gamma$, and let $Q\subset D$ be an annulus with the smooth boundaries Γ and Γ_0. We consider the following mixed boundary problem for the parabolic partial differential equation

$$\left.\begin{array}{l}\partial_t u(x,t)+L_x u(x,t)+f[u(x,t),(\partial_x u(x,t)\cdot\sigma(x)),x,t]=0\\ \partial_x u(x,t)\cdot\gamma(x)=\varphi(x) \qquad (\forall (x,t)\in\Gamma\times[0,T])\\ u(x,T)=\psi(x) \qquad (\forall x\in\bar{D})\end{array}\right\} \tag{1.2}$$

and the mixed Dirichlet-Neumann boundary problem for the elliptic partial differential equation

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$$\left.\begin{array}{l} L_x u(x)+f[u(x),(\partial_x u(x)\cdot\sigma(x)),x]=0 \qquad (\forall x\in Q) \\ \partial_x u(x)\cdot\gamma(x)=\varphi(x) \qquad (\forall x\in\Gamma) \\ u(x)=\psi(x) \qquad (\forall x\in\Gamma_0) \end{array}\right\} \tag{1.3}$$

where $\partial_x=(\partial_1, \partial_2, \cdots, \partial_d)$, and $\sigma(x)$ is a $d\times d$ metric-valued function such that $a(x)=\sigma(x)\cdot\sigma(x)^*$ ($\sigma(x)^*$ is the transpose of $\sigma(x)$.

If the functional f is linear for u and $\partial_x u\cdot\sigma$, for example $f=c(x)u-h(x)$, and $\varphi(x)=\psi(x)=0$, we have known well that, under the usual regularity assumptions, the solution of the problem (1.2) has the following probabilistic representation

$$u(x, t)=E\left[\int_t^T h(X_s)\exp\left(-\int_t^s c(X_r)dr\right)ds\right] \tag{1.4}$$

where X_s is a diffusion process satisfying the following stochastic differential equation (SDE) with reflecting boundary conditions:

$$\left.\begin{array}{l} dX_s=\sigma(X_s)dW_s+b(X_s)ds-\gamma(X_s)d\xi_s \\ d\xi_s=I_\Gamma(X_s)d\xi_s \qquad (X_s\in\bar{D};\ \forall s\geqslant t) \end{array}\right\} \tag{1.5}$$

with the initial conditions $X_t=x$. And the solution of the problem (1.3) has the following representation:

$$u(x)=E\left[\int_0^T h(X_s)\exp\left(-\int_0^s c(X_r)dr\right)ds\right] \tag{1.6}$$

where X_s is the solution of the equation (1.5) with the initial conditions $X_0=x$, and $\tau=\inf\{s\geqslant 0, X_s\in\Gamma_0\}$ is a stopping time (see [1] and [2]).

In this essay, we will consider the quasilinear case. We will show that, under some appropriate conditions for the nonlinear functional $f(u,(\partial_x u\cdot\sigma),\cdot,\cdot)$, the solutions of the quasilinear mixed boundary problems (1.2) or (1.3) can be respectively interpreted as a solution of a backward stochastic differential equations, which is associated with the reflected stochastic differential equation (1.5),

$$-dp_s=f(p_s,q_s,X_s,s)ds-q_s dW_s+\varphi(X_s)d\xi_s \tag{1.7}$$

This kind of backward SDE is first discussed by Pardoux and Peng (see [7] and [8].) In the work of Peng [8], the solutions of quasilinear Dirichlet problems are interpreted as a solution of this kind of backward SDE associated with the classical SDE of Ito's type.

In what follows, we first give a brief discussion on the existence and uniqueness of the solution of the stochastic differential equation with the reflecting boundary conditions (1.5), and establish some important estimations about the solution of this kind of equations.

II. The SDE with Reflecting Boundary Conditions

Let σ and b be $R^{d\times d}$-valued and R^d-valued measurable and bounded functions on R^d respectively, satisfying the condition that there is a constant $K_0>0$

$$|\sigma(x)-\sigma(x')|+|b(x)-b(x')|\leqslant K_0|x-x'| \qquad (\forall x,x'\in R^d) \tag{2.1}$$

Let D be a bounded open set of R^d whose boundary Γ is a C^3-manifold, and let γ be a R^d-valued function on $\bar{D}=D\cup\Gamma$ such that γ is of class $C_b^2(\bar{D})$ and satisfies the following condition:

$$\gamma(x)n(x)\geqslant\delta_1>0 \qquad (\forall x\in\Gamma) \tag{2.2}$$

where δ_1 is a constant and $n(x)$ denotes the exterior unit normal vector at $x\in\Gamma$. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$ be a probability space satisfying the usual hypotheses, and W_t be a R^d-valued standard $(\mathscr{F}_t)$-Wiener process.

Under the assumptions (2.1) and (2.2), it is well-known that, for any $(x,t)\in\bar{D}\times[0,\infty)$, there exists an unique solution (X_s,ξ_s) of the stochastic differential equation (1.5), i. e. there exists a continuous adapted $\bar{D}$-valued process X_s and a continuous adapted non-decreasing process ξ_s satisfying the following integral equation:

$$X_s=x+\int_t^s b(X_r)dr+\int_t^s\sigma(X_r)dW_r-\int_t^s\gamma(X_r)d\xi_r \qquad (\forall s\geqslant t) \tag{2.3}$$

and the boundary condition:

$$\xi_s=\int_t^s I_\Gamma(X_r)d\xi_r \qquad (\forall s\geqslant t) \tag{2.4}$$

(see [1], [2] and [6]). In what follows, we want to prove two important lemma for the solution (X_s,ξ_s), which are used in the next section.

Lemma 2.1 (Lemma 4.1 and Proposition 4.1 in [6])

Under the assumption (2.2), there exists a $d\times d$ symmetric metrix-valued function $e(x)=\{e_{ij}(x)\}_{i,j=1}^d$ such that $e_{ij}(x)\in C_b^2(R^d)$ for any $i,j=1,\ldots,d$ and there are positive constants ν, C_0 such that

$$e(x))\geqslant\nu I_d,\quad \forall x\in R^d;\quad \sum_{j=1}^d e_{ij}(x)\gamma_j(x)=n_i(x) \qquad (\forall 1\leqslant i\leqslant d, x\in\Gamma) \tag{2.5}$$

$$C_0|x-y|^2+\sum_{ij=1}^d e_{ij}(x)(x^i-y^i)\gamma_j(x)\geqslant 0 \qquad (\forall x\in\Gamma, y\in\bar{D}) \tag{2.6}$$

And there exists a function $\Phi(x)\in C_b^2(R^d)$ satisfying that there is a constant $\delta_2>0$ such that

$$\gamma(x)\partial_x\Phi(x)\leqslant-\delta_2 \qquad (\forall x\in\Gamma) \tag{2.7}$$

Lemma 2.2 Assume (2.1) and (2.2) hold, and $(X_s,\ \xi_s)$ is the unique solution of the stochastic differential equation (1.5). Then for any $T>t\geqslant 0$ that are fixed, there exists a constant $C_1>0$ such that for any $s_1,\ s_2\in[t,T]$ and $s_2>s_1$, we have that

$$E|X_{s2}-X_{s1}|^2+E|\xi_{s2}-\xi_{s1}|^2\leqslant C_1|s_2-s_1| \tag{2.8}$$

Proof Let $\lambda>0$ be a constant to be determined. Using (2.5) and (2.7), and applying Ito's formula to the function $F(x)=\exp\{-\lambda\Phi(x)\}x^*e(x)x$, we obtain that

$$\begin{aligned}&E[\exp\{-\lambda\Phi(X_{s2})\}(X_{s2}-X_{s1})^*e(X_{s2})(X_{s2}-X_{s1})]\\ &\leqslant K_1E\left[\int_{s_1}^{s_2}|X_r-X_{s_1}|^2\exp\{-\lambda\Phi(X_r)\}dr\right]+K_2E\left[\int_{s_1}^{s_2}|X_r\right.\\ &\left.-X_{s_1}|^2\exp\{-\lambda\Phi(X_r)\}d\xi_r\right]-\delta_2\nu\lambda E\left[\int_{s_1}^{s_2}|X_r-X_{s_1}|^2\exp\{-\lambda\Phi(X_r)\}d\xi_r\right]\end{aligned}$$

$$-2E\left[\int_{s_1}^{s_2}\exp\{-\lambda\Phi(X_r)\}(X_r-X_{s_1})^*e(X_r)\gamma(X_r)d\xi_r\right]$$

where K_1 and K_2 are constants, and K_2 does not depend on λ. Then using the boundary condition (2.4) and (2.6), and choosing $\lambda=(1/\delta_2\nu)(K_2+2C_0)$, we finally deduce that

$$E\left[\int_{s_1}^{s_2}(X_r-X_{s_1})^*e(X_{s_2})(X_r-X_{s_1})\exp\{-\lambda\Phi(X_r)\}dr\right]$$

$$\leqslant K_1E\left[\int_{s_1}^{s_2}|X_r-X_{s_1}|^2\exp\{-\lambda\Phi(X_r)\}dr\right]$$

or

$$E[|X_{s_2}-X_{s_1}|^2]\leqslant K_1'E\left[\int_{s_1}^{s_2}|X_r-X_{s_1}|^2dr\right]$$

and applying the Gronwall's inequality, we have that

$$E[|X_{s_2}-X_{s_1}|^2]\leqslant C_1'|s_2-s_1|$$

Then since $\gamma(x)$ is bounded in Γ, using similar arguments, we easily obtain that

$$E[|\xi_{s_2}-\xi_{s_1}|^2]\leqslant C_1''|s_2-s_1|$$

Therefore this lemma is proved.

Let $Q\subset D$ be an annulus with the boundaries Γ and Γ_0, and Γ_0 be a C^2-manifold. Let (X_s,ξ_s) be the unique solution of the equation (1.5) with the initial values $X_0=x\in\bar{Q}$ and $\xi_0=0$. Let $\tau=\inf\{s\geqslant0;\ X_s\in\Gamma_0\}$. Then τ is an $(\mathscr{F}_t)$-stopping time.

Lemma 2.3 Assume that there exist constants $\beta>\alpha\geqslant0$ such that

$$\beta I_d\geqslant a(x)\geqslant\alpha I_d\qquad(\forall x\in\bar{Q})\tag{2.9}$$

and there exists a function $\Psi(x)\in C_b^2(R^d)$ and two constants $\delta_3>0$ and $m>0$ such that

$$\partial_x\Psi(x)\cdot\gamma(x)\geqslant\delta_3>0\,(\forall x\in\Gamma);\quad L_x\Psi(x)\geqslant m\qquad(\forall x\in\bar{Q})\tag{2.10}$$

Then, under the assumptions of Lemma 2.2, there exist some constants $\mu>0$ such that

$$E[e^{\mu T}]<\infty \text{ and } E[\xi_T]<\infty\tag{2.11}$$

Proof Let λ and μ be two positive constants to be determined. Applying Ito's formula to the function $F(x,s)=\exp\{-\lambda\Psi(x)+\mu s\}$, we obtain that

$$F(x,t)=E[F(X_{\tau\wedge s},\tau\wedge s)]+E\left[\int_0^{\tau\wedge s}F(X_r,r)(\partial_x\Psi(X_r)\cdot\gamma(X_r))d\xi_r\right]$$

$$-E\left[\int_0^{\tau\wedge s}F(X_r,r)\{\mu+\frac{1}{2}\lambda^2\sum_{ij=1}^{d}a_{ij}(X_r)\partial_i\Psi(X_r)\partial_j\Psi(X_r)-\lambda L_{X_r}\Psi(X_r)\}dr\right.\tag{2.12}$$

We now choose μ such that $0<\mu<m^2/(2\beta M_{\partial\Psi})$, where $M_{\partial\Psi}>0$ is a constant such that $|\partial_x\Psi(x)|\leqslant M_{\partial\Psi}$ for all $x\in\bar{Q}$, and then we have that $m^2-4\mu\left(\frac{1}{2}\beta M_{\partial\Psi}\right)\geqslant0$. Hence, from the assumptions (2.9) and (2.10), we can choose $\lambda>0$ such that, for all $x\in\bar{Q}$

$$\mu+\frac{1}{2}\lambda^2\sum_{ij=1}^{d}a_{ij}(x)\partial_i\Psi(x)\partial_j\Psi(x)-\lambda L_x\Psi(x)\leqslant\mu+\frac{1}{2}\lambda^2\beta M_{\partial\Psi}-\lambda m\leqslant0$$

Thus, we have

$$F(x,t)\geqslant E[F(X_{\tau\wedge s},\tau\wedge s)]\geqslant M_{\lambda}E[\exp[\mu(\tau\wedge s)]]$$

where $M_{\lambda}=\exp(-\lambda M_{\Psi})>0$ and $M_{\Psi}>0$ is a constant such that $|\Psi(x)|\leqslant M_{\Psi}$ for all $x\in\bar{Q}$. Let $s\to\infty$ in the above inequality, and applying Fatou's lemma, we have

$$M_{\lambda}E[\exp[\mu\tau]]\leqslant E[F(X_{\tau},\tau)]\leqslant F(x,t)$$

and so we have that $E[\exp[\mu\tau]]<\infty$.

On the other hand, from (2.12) and for λ and μ chosen above, it can be shown that

$$E\left[\int_t^{\tau\wedge s}F(X_r,r)(\partial_x\Psi(X_r)\cdot\gamma(X_r))d\xi_r\right]\leqslant F(x,t)-E[F(X_{\tau\wedge s},\tau\wedge s)]$$

Let $s\to+\infty$, from Fatou's Lemma, we obtain that

$$E\left[\int_t^{\tau}F(X_r,r)(\partial_x\Psi(X_r)\cdot\gamma(X_r))d\xi_r\right]\leqslant F(x,t)$$

Then from (2.10), it can be shown that $E[\xi_{\tau}]<\infty$. The proof is completed.

III. The Probabilistic Interpretation for Quasilinear Mixed Boundary Problems

Let $\varphi(x)$, $\psi(x)$ be continuous functions on R^d and $f(p,q,x,s)$ be a continuous function on $R\times R^d\times\bar{D}\times[0,\infty)$ satisfying the following conditions that,

(1) there is a constant $C^2>0$ such that, for any p_1, p_2 $\in R$, q_1, $q_2\in R^d$ and $(x,s)\in\bar{D}\times[0,\infty)$.

$$|f(p_1,q_1,x,s)-f(p_2,q_2,x,s)|\leqslant C_2(|p_1-p_2|+|q_1-q_2|) \tag{3.1}$$

(2) there is a constant $\delta_4>0$ such that, for any $p,p_1\in R,(q,x,s)\in R^d\times\bar{D}\times[0,+\infty)$

$$p_1(f(p+p_1,q,x,s)-f(p,q,x,s))\leqslant-\delta_4 p_1^2 \tag{3.2}$$

Let $\theta\geqslant t\geqslant 0$ be a $(\mathscr{F}_t)$-stopping time. We consider the following backward stochastic differential equation

$$\left.\begin{aligned}&-dp_s=f(p_s,q_s,X_s,s)ds\quad q_sdW_s+\varphi(X_s)d\xi_s\qquad((s\in[t,\theta]))\\&p_\theta=\psi(X_\theta),\end{aligned}\right\} \tag{3.3}$$

where (X_s,ξ_s) is the unique solution of the forward SDE (2.3) with reflecting boundary conditions (2.4) with the initial conditions that $X_t=x\in\bar{D}$ and $\xi_t=0$. We say that a pair of processes (p_s,q_s) is a solution of the backward SDE (3.3) if p_s is a $(\mathscr{F}_s)$-adapted continuous process and q_s is a R^d-valued process, and they both satisfy the following integral equation:

$$p_s=\psi(X_\theta)+\int_s^\theta f(p_r,q_r,X_r,r)dr-\int_s^\theta q_rdW_r+\int_s^\theta\varphi(X_r)d\xi_r,\qquad(s\in[t,\theta]) \tag{3.4}$$

Theorem 3.1 Let $\theta=T>t$. Assume that σ and b are bounded continuous functions satisfying (2.1), $\gamma\in C_b^2(\bar{D})$ satisfying (2.2), φ,ψ and f are continuous functions, and f satisfies (3.1) and (3.2). Then the backward stochastic differential equation (3.3) has a unique $(\mathscr{F}_s)$-adapted solution (p_s,q_s) . Moreover, (p_s,q_s) also satisfies that

$$E\left[\int_t^T |p_r|^2 dr\right]<\infty, \quad E\left[\int_t^T |q_r|^2 dr\right]<\infty \tag{3.5}$$

Proof Since φ, ψ and f are continuous and $\bar{D}$ is bounded, φ, ψ and $f(p,q,\cdot,\cdot)$ are bounded on $\bar{D}$ and $\bar{D}\times[0,T]$. Therefore from Lemma 2.2, we have that

$$E[|\psi(X_T)|^2]+E\left[\int_0^T |f(0,0,X_r,r)|^2 dr\right]<\infty \tag{3.6}$$

and

$$E\left[\int_0^T |\varphi(X_r)|^2 d\xi_r\right]<\infty \tag{3.7}$$

From these estimations and using (3.1) and (3.2), using the same arguments as the proof of Theorem 2.2 in [8], we can prove this theorem. Here we omit the details.

Let τ be the stopping time defined in the Section II. Then, as in Theorem 3.1, we can prove the following theorem by using Lemma 2.2 and Lemma 2.3.

Theorem 3.2 Let $\theta=\tau$ and $t=0$. Assume the same conditions as in Lemma 2.3 and Theorem 3.1 are satisfied. Then the backward stochastic differential equation (3.3) has a unique $(\mathscr{F}_s)$-adapted solution (p_s,q_s). Moreover, (p_s,q_s) satisfies the following conditions that

$$E\left[\int_0^\tau |p_r|^2 dr\right]<\infty, \quad E\left[\int_0^\tau |q_r|^2 dr\right]<\infty \tag{3.8}$$

In what follows, we will show that the solution of the quasilinear mixed Dirichlet-Neumann boundary problem (1.2) or the mixed Dirichlet-Neumann boundary problem (1.3) can be respectively interpreted by the solution of the backward SDE (3.3), which is given in Theorem 3.1 and Theorem 3.2 respectivly.

Theorem 3.3 Suppose that the mixed boundary problem for the parabolic partial differential equation:

$$\left.\begin{array}{l} \partial_t u(x,t)+L_x u(x,t)+f[u(x,t),(\partial_x u(x,t)\cdot\sigma(x)),x,t]=0 \\ \partial_x u(x,t)\cdot\gamma(x)=\varphi(x) \qquad (\forall (x,t)\in\Gamma\times[0,T]) \\ u(x,T)=\psi(x) \qquad (\forall x\in\bar{D}) \end{array}\right\} \tag{3.9}$$

has a solution $u(x,t)\in C_b^2(\bar{D}\times[0,\infty))$. Then, under the same assumptions as those in Theorem 3.1, $u(x,t)$ has the following interpretation:

$$u(x,t)=p_t \tag{3.10}$$

where p_t is a stochastic process determined uniquely by the equation (3.3) for $\theta=T$ in Theorem 3.1.

Proof Applying Ito's formula to the diffusion process X_t given in (2.3) for the solution $u(x,t)$ of the equation (3.3), we have that

$$u(X_T,T)-u(X_s,s)=\int_s^T (\partial_r u(X_r,r)+L_{X_r}u(X_r,r))dr$$

$$-\int_s^T (\partial_x u(X_r,r)\cdot\gamma(X_r))d\xi_r+\int_s^T (\partial_x u(X_r,r)\cdot\sigma(X_r))dW_r$$

Since $u(x, t)$ satisfies the mixed boundary problem (3.9), and (X_s,ξ_s) satisfies the reflecting boundary conditions (2.4), we have that

$$u(X_s,s)=\psi(X_T)+\int_s^T f(u(X_r,r),v(X_r,r),X_r,r)dr$$
$$+\int_s^T \varphi(X_r)d\xi_r-\int_r^T(\partial_x u(X_r,r)\cdot\sigma(X_r))dW_r$$

where $v(X_s,s)=(\partial_x u(X_s,s)\cdot\sigma(X_s))$. Hence $(u(X_s,s),v(X_s,s))$ is the unique solution of backward SDE (3.3). It follows that

$$u(x,\ t)=p_t \tag{3.11}$$

the proof is completed.

Theorem 3.4 Suppose that the mixed Dirichlet-Neumann boundary problem for the elliptic partial differential equation:

$$\left.\begin{aligned}&L_x u(x)+f(u(x),(\partial_x u(x)\cdot\sigma(x)),x)=0 \qquad (\forall x\in Q)\\&\partial_x u(x)\cdot\gamma(x)=\varphi(x) \qquad (\forall x\in\Gamma)\\&u(x)=\psi(x) \qquad (\forall x\in\Gamma_0)\end{aligned}\right\} \tag{3.12}$$

has a solution $u(x)\in C_b^2(\bar{D})$. Then, under the same assumptions as those in Theorem 3.2, $u(x)$ has the following interpretation:

$$u(x)=p_0 \tag{3.13}$$

where p_0 is determined uniquely by the equation (3.3) for $\theta=\tau$ and $t=0$ in Theorem 3.2.

Proof Applying Ito's formula to the diffusion process given in the equation (2.3) for the solution $u(x)$ of the problem (3.12), we have that

$$u(X_\tau)-u(X_0)=\int_0^\tau LX_r u(X_r)dr$$
$$-\int_0^\tau(\partial_x u(X_r)\cdot\gamma(X_r))d\xi_r+\int_0^\tau(\partial_x u(X_r)\cdot\sigma(X_r))dW_r$$

Since $u(x)$ solves (3.12) and (X_s,ξ_s) satisfies the reflecting boundary conditions (2.4), we have that

$$u(X_s)=\psi(X_\tau)+\int_s^\tau f(u(X_r),v(X_r),X_r)dr$$
$$+\int_s^\tau \varphi(X_r)d\xi_r-\int_s^\tau(\partial_x u(X_r)\cdot\sigma(X_r))dW_r$$

where $v(X_s)=(\partial_x u(X_s)\cdot\sigma(X_s))$. Hence $(u(X_s),v(X_s))$ is the unique solution of (3.3). It follows that

$$u(x)=p_0 \tag{3.14}$$

the proof is completed.

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