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THREE-DIMENSIONAL ANALYSIS ON PLATES

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Abstract

The displacements of the plate are assumed appropriately to derive the solutions of the 3-D Navier equations. And the conditions on the plate's surface are investigated. In the examples, the boundary-value problems of the plate are solved by applying the Navier-equation's solutions and their closed-form solutions are obtained. The results formulated in the present paper satisfy exactly the governing equations and can reflect precisely the boundary effects of complicated distributions on the edge of plates

Key words plate, thick plate, displacement method, boundary-value problems, Navier equation

I. Introduction

Generally, as a three-dimensional (3-D) solid, the plate should be analysed with the thick plate theory. The thick plate theory is multifarious: A customary thick theory originates from various modifications of the thin plate theory, which is based on Kirchhoff-Love hypothesis, with shear strains and trasverse normal strains[1]. The relatively typical and the most extensively applied in numerical computations in recent are Reissner theory[2] and Mindlin theory[3].

A great number of the plate problems have never been solved satisfactoryly. Hence new theories and methods are invented successively. It is worth noticing that the theories developed in the rescent years counting in the higher-order effects of the shear strains have improved enormously the computational precision[4].

The methods mentioned above can constantly only obtain proximal results. In order to calculate the plate precisely, the 3-D boundary-value problems should be solved.

Navier equations are the governing equations in the boundary-problems that describe the elastic solids with displacements, and it is of significance find their general solutions. Works [5~7] sumed up the general solutions. Paper [8] discussed the mechanical method to constitute the general solutions. All these solutions are successful in the applications to solve the problems of the elastic half-space media and contact problems, but it is difficult to use them to solve plate problems.

The present paper has found the solutions, suitable to the plate problems, of Navier equations and explained the solving process by using the solutions of Navier equation through solving some practical problems.

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II. Solutions of Navier Equations

As is shown in Fig. 1, $Oxyz$ is the Cartesian rectangular coordinate system. Its Oxy coordinate plane coincides with the upper plate-surface.

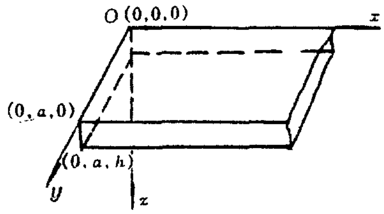

Fig. 1 Plate and its coordinate system

The Navier equation of the isotropic, linear ealstic solid of the small strain and of no body force is

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ji}=0 \qquad (2.1)$$

where λ and μ are Lamé constants; $i=1, 2, 3$; $j=1, 2, 3$; the summation convention is maintained when indexes coincide; u_1, u_2 and u_3 are the components along the x, y and z axes respectively of the displacement of a particle within the plate and are denoted as u, v and w respectively.

$$\begin{aligned} u(x,y,z)&=-f(z)\frac{\partial W(x,y)}{\partial x} \\ v(x,y,z)&=-g(z)\frac{\partial W(x,y)}{\partial y} \\ w(x,y,z)&=\varphi(z)W(x,y) \end{aligned} \qquad (2.2)$$

Here: u, v and w are the components, which are along x, y and z axes respectively, of the displacement of a particle within the plate respectively; $f(z)$, $g(z)$ and $\varphi(z)$ are thickwise distributing functions, which are determined by the differential equations; $W(x, y)$ is a distributing function of the deflection, which is assumed to be

$$W(x,y)=\overline{W}\exp(\alpha x+\theta y) \qquad (2.3)$$

where $\overline{W}$, α and θ are arbitrary constants, whose values are related to the boundary conditions.

The stresses can be expressed as

$$\sigma_{zz}=(\lambda+2\mu)\frac{d\varphi}{dz}W-\lambda\left(f\frac{\partial^2 W}{\partial x^2}+g\frac{\partial^2 W}{\partial y^2}\right) \qquad (2.4a)$$

$$\sigma_{zx}=\mu\left(\varphi-\frac{df}{dz}\right)\frac{\partial W}{\partial x} \qquad (2.4b)$$

$$\sigma_{zy}=\mu\left(\varphi-\frac{dg}{dz}\right)\frac{\partial W}{\partial y} \qquad (2.4c)$$

$$\sigma_{xx}=-(\lambda+2\mu)f\frac{\partial^2 W}{\partial x^2}+\lambda\left(-g\frac{\partial^2 W}{\partial y^2}+\frac{d\varphi}{\partial z}W\right) \qquad (2.4d)$$

$$\sigma_{xy}=-\mu(f+g)\frac{\partial^2 W}{\partial x\partial y} \qquad (2.4e)$$

$$\sigma_{yy}=-(\lambda+2\mu)g\frac{\partial^2 W}{\partial y^2}-\lambda f\frac{\partial^2 W}{\partial x^2}+\lambda\frac{d\varphi}{dz}W \qquad (2.4f)$$

Substituting (2.2) and (2.3) into (2.1) yields

$$\left\{-[(\lambda+2\mu)\alpha^2+\mu\theta^2]f(z)-\mu\frac{d^2f(z)}{dz^2}+(\lambda+\mu)\left[-\theta^2g(z)+\frac{d\varphi(z)}{dz}\right]\right\}\alpha W(x,y)=0 \tag{2.5a}$$

$$\left\{-[(\lambda+2\mu)\theta^2+\mu\alpha^2]g(z)-\mu\frac{d^2g(z)}{dz^2}+(\lambda+\mu)\left[-\alpha^2f(z)+\frac{d\varphi(z)}{dz}\right]\right\}\theta W(x,y)=0 \tag{2.5b}$$

$$\left\{(\lambda+2\mu)\frac{d^2\varphi(z)}{dz^2}+\mu k^2\varphi(z)-(\lambda+\mu)\left[\alpha^2\frac{df(z)}{dz}+\theta^2\frac{dg(z)}{dz}\right]\right\}W(x,y)=0 \tag{2.5c}$$

where

$$k^2=\alpha^2+\theta^2 \tag{2.6}$$

The system of differential equations is solved as follows with respect to two cases $k=0$ and $k\neq 0$.

1. $k=0$

(1) $\alpha=\theta=0$

In this case, it can be known from expression (2.3) that $W(x,y)=\overline{W}=$ constant. Substituting it into the system of differential equations (2.5) leads to $\frac{d^2\varphi(z)}{dz^2}=0$. Thus, the displacements are

$$u(x,y,z)=v(x,y,z)\equiv 0,\quad w(x,y,z)=A_0z+B_0 \tag{2.7}$$

where A_0 and B_0 are arbitrary constants.

(2) α and θ being not both zero

The solutions of the system of differential equations (2.5) are

$$f(z)=-\frac{(\lambda+\mu)\alpha^2A_1z^3}{6(\lambda+2\mu)}+\frac{C}{2}z^2+Dz+E \tag{2.8a}$$

$$g(z)=-\frac{(\lambda+\mu)\alpha^2A_1z^3}{6(\lambda+2\mu)}+\frac{C}{2}z^2+(D-A_1)z+E-B_1 \tag{2.8b}$$

$$\varphi(z)=\frac{(\lambda+\mu)\alpha^2A_1z^2}{2(\lambda+2\mu)}+\left(\alpha^2B_1+\frac{\mu C}{\lambda+\mu}\right)z+F \tag{2.8c}$$

where A_1, B_1, C, D, E, F are arbitrary constants.

2. $k\neq 0$

In this case, it can be found from the system of differential equations (2.5) that

$$f(z)=(C_1+C_2z)\cos kz+(C_3+C_4z)\sin kz \tag{2.9a}$$

$$g(z)=(-A+C_1+C_2z)\cos kz+(-B+C_3+C_4z)\sin kz \tag{2.9b}$$

$$\begin{aligned}\varphi(z)=-\frac{1}{k^2(\lambda+\mu)}\{&A(\lambda+\mu)k\theta^2\sin kz-B(\lambda+\mu)k\theta^2\cos kz\\&-C_1(\lambda+\mu)k^3\sin kz-C_2[(\lambda+3\mu)k^2\cos kz+(\lambda+\mu)k^3z\sin kz]\\&+C_3(\lambda+\mu)k^3\cos kz+C_4[(\lambda+\mu)k^3z\cos kz-(\lambda+3\mu)k^2\sin kz]\}\end{aligned} \tag{2.9c}$$

Here A, B, C_1, C_2, C_3, and C_4 are arbitrary constants.

The constant coefficients of the equations (2.7)~(2.9) are determined by the plate-surface conditions and the edge conditions.

III. Homogeneous Plate-Surface Conditions

At first, let us consider the problems of the plate with free plat-surfaces. As is shown in Fig. 1, the thickness of the plate is h.

The plate-surface conditions are: $\sigma_{zz}=\sigma_{zx}=\sigma_{zy}=0$, when $z=0$ or $z=h$. Using (2.4), in both of the cases $z=0$ and $z=h$, we have

$$(\lambda+\mu)\frac{d\varphi}{dz}W-\lambda\left[f\frac{\partial^2 W}{\partial x^2}+g\frac{\partial^2 W}{\partial y^2}\right]=0 \tag{3.1a}$$

$$\mu\left(\varphi-\frac{df}{dz}\right)\frac{\partial W}{\partial x}=0 \tag{3.1b}$$

$$\mu\left(\varphi-\frac{dg}{dz}\right)\frac{\partial W}{\partial y}=0 \tag{3.1c}$$

It follows that the situations corresponding to the different values of k are discussed

1. $k=0$

(1) $\alpha=\theta=0$

Substituting (2.7) into (3.1) results in

$$u=v\equiv 0,\quad w=\text{const.}$$

In this case, there is only rigid displacement along the direction perpendicular to the plate-surface. This is not interesting, so the latter part of this paper will provide that not both of α and θ are zero.

(2) α and θ being not both zero

Substituting $\theta^2=-\alpha^2$ and (2.8) into the system of equations (3.1) and getting the coefficients, we have

$$f(z)=g(z)=Dz+E,\quad \varphi(z)=D \tag{3.2}$$

It can be concluded from the substitution of the above expression into (2.4a)~(2.4c) that above expressions imply that the shear strains and transverse normal strains are zero. i. e.

$$\gamma_{yz}(x,y,z)=\gamma_{xz}(x,y,z)=\varepsilon_z(x,y,z)\equiv 0$$

If $E/D=-h/2$, in view of (3.2) and (2.2), the in-plane displacements of the particles on the middle plane ($z=h/2$) of the plate are zero, while the in-plane displacements of the other particles within the plate vary linearly along the thickness. That is, the plate bends with the middle plane as the neutral surface, and the transverse sections remain plane after the deformation. Therefore, when $E/D=-h/2$, Kirchhoff hypothesis of the thin plate of small deflection holds precisely:

2. $k\neq 0$

Substituting (2.9) into (3.1) and arranging the results, we can obtain the system of the linear algebraic equations:

$$\begin{bmatrix} -(\lambda+\mu)\theta^2\cos kh & -(\lambda+\mu)\theta^2\sin kh & (\lambda+\mu)k^2\cos kh \\ -(\lambda+\mu)\theta^2 & 0 & (\lambda+\mu)k^2 \end{bmatrix}$$

$$
\begin{array}{lll}
\frac{(\lambda+\mu)}{2}\theta^2 k\sin kh & -\frac{(\lambda+\mu)}{2}\theta^2 k\cos kh & -(\lambda+\mu)k^3\sin kh \\
0 & -\frac{\lambda+\mu}{2}\theta^2 k & 0 \\
\frac{(\lambda+\mu)}{2}(k\theta^2+k^3)\sin kh & -\frac{(\lambda+\mu)}{2}(\theta^2 k+k^3)\cos kh & -(\lambda+\mu)k^3\sin kh \\
0 & -\frac{(\lambda+\mu)}{2}(\theta^2 k+k^3) & 0
\end{array}
$$

$$
\begin{array}{ll}
(\lambda+\mu)k^2 h\cos kh-(\lambda+2\mu)k\sin kh & (\lambda+\mu)k^2\sin kh \\
0 & 0 \\
-\mu k^2\cos kh-(\lambda+\mu)k^3 h\sin kh & (\lambda+\mu)k^3\cos kh \\
-\mu k^2 & (\lambda+\mu)k^3 \\
-\mu k^2\cos kh-(\lambda+\mu)k^3 h\sin kh & (\lambda+\mu)k^3\cos kh \\
-\mu k^2 & (\lambda+\mu)k^3
\end{array}
$$

$$
\left.\begin{array}{c}
(\lambda+\mu)k^2 h\sin kh+(\lambda+2\mu)k\cos kh \\
(\lambda+2\mu)k \\
-\mu k^2\sin kh+(\lambda+\mu)k^3 h\cos kh \\
0 \\
-\mu k^2\sin kh+(\lambda+\mu)k^3 h\cos kh \\
0
\end{array}\right]
\begin{Bmatrix} A \\ B \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\tag{3.3}
$$

The system of equations (3.3) has non-trivial solutions, if and only if the determinant is zero. So

$$(\lambda+\mu)^4 k^8\sin kh[-(kh)^2+\sin^2 kh]=0$$

Its roots are $k=\frac{n\pi}{h}(n=\pm 1,\pm 2,\cdots)$. The integer n can take the values of natural number, i. e.

$$k=\frac{n\pi}{h} \qquad (n=1,2,\cdots) \tag{3.4}$$

Substituting (3.4) into (3.3), we find

$$C_1=\left(\frac{\theta}{k}\right)^2 A,\quad B=C_2=C_3=C_4=0 \tag{3.5}$$

Substituting the expressions (3.5) into expressions (2.9) gives

$$f(z)=\left(\frac{\theta}{k}\right)^2 A\cos kz$$

$$g(z)=-\left(\frac{\alpha}{k}\right)^2 A\cos kz$$

$$\varphi(z)\equiv 0$$

It could be seen that, when $k=\frac{n\pi}{h}$, the plate has only the in-plane deformation along the direction parallel to the plate-surface. However, when $A\neq 0$, only the component $\sigma_{zz}=0$ among the six independent components of the stress.

Summing up, the general term of the independent column series of the displacement column $\{u \quad v \quad w\}^T$ are

$$\left\{\begin{matrix} -z\dfrac{\partial W_{m0}(x,y)}{\partial x} \\ -z\dfrac{\partial W_{m0}(x,y)}{\partial y} \\ W_{m0}(x,y)\end{matrix}\right\};\quad \left\{\begin{matrix} -\dfrac{\partial W_{m0}(x,y)}{\partial x} \\ -\dfrac{\partial W_{m0}(x,y)}{\partial y} \\ 0\end{matrix}\right\},\quad \left\{\begin{matrix} -\left(\dfrac{\theta_{mn}}{k_n}\right)^2\cos k_n z\,\dfrac{\partial W_{mn}(x,y)}{\partial x} \\ \left(\dfrac{\alpha_{mn}}{k_n}\right)^2\cos k_n z\,\dfrac{\partial W_{mn}(x,y)}{\partial y} \\ 0\end{matrix}\right\}$$

$$(m=1,2,\cdots;\ n=1,2,\cdots) \qquad (3.7)$$

where $W_{m0}(x,y)$ can be chosen in

$$\cos\alpha_m x\,\mathrm{ch}\,\alpha_m y,\ \cos\alpha_m x\,\mathrm{sh}\,\alpha_m y,\ \sin\alpha_m x\,\mathrm{ch}\,\alpha_m y,\ \sin\alpha_m x\,\mathrm{sh}\,\alpha_m y$$

and $W_{mn}(x,y)$ can be chosen in

$$\cos\beta_{mn}x\,\mathrm{ch}\,\theta_{mn}y,\cos\beta_{mn}x\,\mathrm{sh}\,\theta_{mn}y,\sin\beta_m x\,\mathrm{ch}\,\theta_{mn}y,\ \sin\beta_{mn}x\,\mathrm{sh}\,\theta_{mn}y$$

Here: $\beta_{mn}^2+\theta_{mn}^2=k_n^2=\left(\frac{n\pi}{h}\right)^2$; $\alpha_m,\beta_{mn},\theta_{mn}$ can take complex values.

In view of (3.7), the displacement solution satisfying the homogeneous conditions (3.1) on the surface of the plate can be given by

$$u(x,y,z)=-\sum_{m=1}^{\infty}A_{m0}z\frac{\partial W_{m0}(x,y)}{\partial x}-\sum_{m=1}^{\infty}B_{m0}\frac{\partial W_{m0}(x,y)}{\partial x}$$
$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left(\frac{\theta_{mn}}{k_n}\right)^2\cos k_n z\frac{\partial W_{mn}(x,y)}{\partial x}$$

$$v(x,y,z)=-\sum_{m=1}^{\infty}A_{m0}z\frac{\partial W_{m0}(x,y)}{\partial y}-\sum_{m=1}^{\infty}B_{m0}\frac{\partial W_{m0}(x,y)}{\partial y}$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left(\frac{\alpha_{mn}}{k_n}\right)^2\cos k_n z\frac{\partial W_{mn}(x,y)}{\partial y}$$
$$w(x,y,z)=\sum_{m=1}^{\infty}A_{m0}W_{m0}(x,y) \qquad (3.8)$$

The arbitrary constants α_m, β_{mn}, θ_{mn}, A_{m0}, B_{m0} and C_{mn} should be determined further by the edge conditions. For the purpose of this determination, the given functions of the distributed forces on the edges are generally expanded into double trigonometric series. Then β_{mn} or θ_{mn} are usually taken as imaginary values to transfer hyperbolic functions into trigonometric functions. In such a case, α or θ in (3.6) should take imaginary value.

IV. Nonhomogeneous Plate-Surface Conditions

Next, we consider the plate problems with loads on the surface of the plate.

We will discuss only the common situations of the plate-surface loads. That is, the plate-surface loads are exerted normally to the plate-surface. The investigation herein suits the loads arbitrarily distributed over the plate-surface.

The stress conditions on the plate-surfaces are

$$z=0:\quad \sigma_{zz}=-q(x,y),\quad \sigma_{zx}=\sigma_{zy}=0 \tag{4.1}$$

$$z=h:\quad \sigma_{zz}=\sigma_{zx}=\sigma_{zy}=0 \tag{4.2}$$

The function $q(x,y)$ may be an arbitrary function of x and y, except the confinement that it must be absolutely integral in the area of the plate-surface. According to Fourier theory, $q(x,y)$ can be expanded into one of the sine-sine series, sine-cosine series, cosine-sine series and cosine-cosine series. Nothing of the generality is lost when we discuss the sine-sine series. Then, when the area of the plate-surface is a rectangular of $a\times b$, we have

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4.3}$$

where

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(r,s)\sin\frac{m\pi r}{a}\sin\frac{n\pi s}{b}drds \tag{4.4}$$

1. Distributing functions of deflection

Comparing (4.1) with (4.3) makes it clear that in (2.3) α and θ can be replaced with $i\alpha$ and $i\theta$ respectively. The parameters α and θ are real in $i\alpha$ and $i\theta$. Appropriate combination results in

$$W(x,y)=\sin\alpha x\sin\theta y,\quad \sin\alpha x\cos\theta y,\quad \cos\alpha x\sin\theta y,\quad \cos\alpha x\cos\theta y \tag{4.5}$$

where $W(x,y)=\sin\alpha x\sin\theta y$ matches the expression (4.3) well, and it is known from (2.2) that such a solution suits solving the problems of simply supported rectangular plate. The other else terms can be chosen for the other else support conditions.

2. Thickwise distribution function

Here, the discussion is confined for $a\times b$ rectangular plate with all the four edges being simply supported.

In view of (2.6), when α and θ in (2.3) are replaced with $i\alpha$ and $i\theta$, k should also be replaced with ik. In such a case, (2.3) becomes (4.5), and (2.9) should become

$$f(z)=(C_1+C_2z)\mathrm{ch}kz+(C_3+C_4z)\mathrm{sh}kz \tag{4.6a}$$

$$g(z)=(-A+C_1+C_2z)\mathrm{ch}kz+(-B+C_3+C_4z)\mathrm{sh}kz \tag{4.6b}$$

$$\begin{aligned}\varphi(z)=\frac{1}{k^2(\lambda+\mu)}\{&A(\lambda+\mu)k\theta^2\mathrm{sh}kz+B(\lambda+\mu)k\theta^2\mathrm{ch}kz\\&-C_1(\lambda+\mu)k^3\mathrm{sh}kz+C_2[(\lambda+3\mu)k^2\mathrm{ch}kz-(\lambda+\mu)k^3z\mathrm{sh}kz]\\&-C_3(\lambda+\mu)k^3\mathrm{ch}kz+C_4[-(\lambda+\mu)k^3z\mathrm{ch}kz+(\lambda+3\mu)k^2\mathrm{sh}kz]\}\end{aligned} \tag{4.6c}$$

Corresponding to the general load term q_{mn}, the parameters α, θ and k are $\frac{m\pi}{a}$, $\frac{n\pi}{b}$, and $\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$ respectively. Likewise the procedure to the situation of the homogeneous plate-surface condition is repeated for the establishment of the system of linear algebraic equations about A_{mn}, B_{mn}, C_{1mn}, C_{2mn}, C_{3mn}, C_{4mn}. which are found and substituted into (4.6) to give

$$f_{mn}(z)=g_{mn}(z)=\frac{q_{mn}}{2k_{mn}}\cdot\frac{1}{(k_{mn}h)^2-\mathrm{sh}^2k_{mn}h}$$
$$\cdot\left\{\left[\frac{(\lambda+\mu)(k_{mn}h)^2+\mu\mathrm{sh}^2(k_{mn}h)}{\mu(\lambda+\mu)k_{mn}}-\frac{k_{mn}h+\mathrm{ch}k_{mn}h\mathrm{sh}k_{mn}h}{\mu}z\right]\mathrm{ch}k_{mn}h\right.$$
$$\left.+\left[-\frac{k_{mn}h+\mathrm{ch}k_{mn}h\mathrm{sh}\,k_{mn}h}{(\lambda+\mu)k_{mn}}+\frac{\mathrm{sh}^2k_{mn}h}{\mu}z\right]\mathrm{sh}k_{mn}z\right\}$$
$$\begin{pmatrix}m=1,2,3,\cdots\\ n=1,2,3,\cdots\end{pmatrix} \tag{4.7a、b}$$

$$\varphi_{mn}(z)=\frac{q_{mn}}{2(\lambda+\mu)\mu k_{mn}}\cdot\frac{1}{(k_{mn}h)^2-\mathrm{sh}k_{mn}h}\{[-(\lambda+2\mu)(k_{mn}h$$
$$+\mathrm{ch}k_{mn}h\mathrm{sh}k_{mn}h)-(\lambda+\mu)k_{mn}z\mathrm{sh}^2k_{mn}h]\mathrm{ch}k_{mn}z$$
$$+[-(\lambda+\mu)(k_{mn}h)^2+(\lambda+2\mu)\mathrm{sh}^2k_{mn}h+(\lambda+\mu)k_{mn}z(k_{mn}h$$
$$-\mathrm{ch}k_{mn}h\mathrm{sh}k_{mn}h)]\mathrm{sh}k_{mn}z\}\quad\begin{pmatrix}m=1,2,3,\cdots\\ n=1,2,3,\cdots\end{pmatrix} \tag{4.7c}$$

Consequently, the displacement of the simply supported rectangular plate can be expressed as

$$u(x,y,z)=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}f_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4.8a}$$

$$v(x,y,z)=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\pi}{b}f_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{4.8b}$$

$$w(x,y,z)=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\varphi_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4.8c}$$

This is similar to Navier method of the classical plate theory. The difference is that the solution here can satisfy Navier equations rigorously, but Navier method can not. Nevertheless, it can be seen from the example analysis in the next section that, when the thickness of the plate tends zero, the solution of the present paper tends the same result as that of the solution of Navier method.

V. Examples

Here, in order to explain the procedure of the 3-D analysis of the plate problems, two types of examples, the semi-infinite plates without plate-surface load and the simply supported rectangular plates with plate-surface load, are provided.

1. Semi-infinite plates without plate-surface load

The semi-infinite plate takes up the space: $0\leqslant x<\infty$, $-\infty<y<\infty$, $0\leqslant z\leqslant h$. No load is exerted on the surfaces of the plate. At the location of $x\to\infty$, displacements vanish, i. e.

$$x\to\infty:\ u\to 0,\ v\to 0,\ w\to 0 \tag{5.1}$$

At the edge of $x=0$, two typical cases of distributed force are investigated respectively as follows.

(1) Linearly varying normal stress

The boundary conditions at $x=0$ are set as

$$x=0;\quad \sigma_{xx}=\frac{12M_0}{h^3}\left(z-\frac{h}{2}\right)\sin\frac{m\pi y}{a} \tag{5.2a}$$

$$\sigma_{xy}=-\frac{12M_0}{h^3}\left(z-\frac{h}{2}\right)\cos\frac{m\pi y}{a} \tag{5.2b}$$

where M_0 is the bending moment per unit length of plate, and takes a given value; m and a also take given values. When m is an integer, the semi-infinite plate can be taken as a semi-infinite strip with a width of a along y-axis and with two opposite edges being fixpinned.

According to the boundary condition (5.1) and (5.2), noticing $\theta^2=-\alpha^2$, and using (3.2) and (2.3), we can constitute the solution. Thus, it is reasonable that

$$f(z)=g(z)=Dz+E$$

$$\varphi(z)=D$$

$$W(x,y)=\exp\left(-\frac{m\pi x}{a}\right)\sin\frac{m\pi y}{a} \tag{5.3}$$

Substituting (5.3) into (2.4d) yields

$$\sigma_{zz}=-2\mu\left(\frac{m\pi}{a}\right)^2(Dz+E)\exp\left(-\frac{m\pi}{a}x\right)\sin\frac{m\pi y}{a}$$

And then substituting the above expression into (5.2) gives

$$D=-\frac{6M_0}{\mu h^3}\left(\frac{a}{m\pi}\right)^2$$

$$E=\frac{3M_0}{\mu h^2}\left(\frac{a}{m\pi}\right)^2$$

Substituting the above expressions and (5.3) into (2.2), we obtain the displacement solution

$$u(x,y,z)=-\frac{6M_0}{\lambda h^3}\frac{a}{m\pi}\left(z-\frac{h}{2}\right)\exp\left(-\frac{m\pi x}{a}\right)\sin\frac{m\pi y}{a} \tag{5.4a}$$

$$v(x,y,z)=\frac{6M_0}{\mu h^3}\frac{a}{m\pi}\left(z-\frac{h}{2}\right)\exp\left(-\frac{m\pi x}{a}\right)\cos\frac{m\pi y}{a} \tag{5.4b}$$

$$w(x,y,z)=-\frac{6M_0}{\mu h^3}\left(\frac{a}{m\pi}\right)^2\exp\left(-\frac{m\pi x}{a}\right)\sin\frac{m\pi y}{a} \tag{5.4c}$$

It can be concluded from putting $z=h/2$ in (5.4a) and (5.4b) that the in-plane displacements in the middle plane are zero. This means that in the present case the middle plane of plate is the neutral surface of the bending deformation.

As can be shown, the same result can also be obtained with Levy-method of the small deflection theory of the thin plate. Hence, no matter how thick the plate is, the characteristics of the plate's bending deformation will coincide with Kirchhoff hypothesis when no transverse shear and no in-plane traction (compression) are acted.

(2) Square-wavelike varying normal stress

Suppose that the plate at $x=0$ has the following boundary conditions:
As $x=0$, $-\infty<y<\infty$:

$$\sigma_{zz}=\begin{cases}-\sigma\sin\dfrac{m\pi y}{a} & \left(0<z<\dfrac{h}{2}\right)\\ \sigma\sin\dfrac{m\pi y}{a} & \left(\dfrac{h}{2}<z<h\right)\end{cases} \tag{5.5}$$

$$M_{zy}=H_0\cos\frac{m\pi y}{a} \tag{5.6}$$

where H, σ and a have definite values; m is a natural number; and M_{zy} is a torque per unit width of plate, i. e.

$$M_{zy}=\int_0^h\left(z-\frac{h}{2}\right)\sigma_{zy}|_{x=0}dz$$

When the plastic region of the body which is connected to the plate has completely developed, the boundary condition takes the form described by (5.5).

Now, we use (3.8) to find the solution. The displacements are then expressed as

$$\begin{aligned}u(x,y,z)=&-(Dz+E)\frac{m\pi}{a}\exp\left(-\frac{m\pi x}{a}\right)\sin\frac{m\pi y}{a}\\&-\sum_{n=1}^{\infty}A_n\left(\frac{mh}{na}\right)^2\pi\sqrt{\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2}\cos\frac{n\pi z}{h}\\&\cdot\exp\left[-\pi\sqrt{\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2}x\right]\sin\frac{m\pi y}{a}\end{aligned} \tag{5.7a}$$

$$\begin{aligned}v(x,y,z)=&-(Dz+E)\frac{m\pi}{a}\exp\left(-\frac{m\pi x}{a}\right)\cos\frac{m\pi y}{a}\\&+\sum_{n=1}^{\infty}A_n\frac{m\pi}{a}\cdot\frac{\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2}{(n/h)^2}\cos\frac{n\pi z}{h}\\&\cdot\exp\left[-\pi\sqrt{\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2}x\right]\cos\frac{m\pi y}{a}\end{aligned} \tag{5.7b}$$

$$w(x,y,z)=D\exp\left(-\frac{m\pi x}{a}\right)\sin\frac{m\pi y}{a} \tag{5.7c}$$

where D, E and A_n are unknown constants, which are determined as follows.

Substituting (5.7), (2.4d) into (5.5) and expanding the right side of (5.5) into cosine series with regard to x, we find

$$\begin{aligned}&2E+Dh=0,\quad A_n=0\qquad(n=2,\ 4,\ 6,\ \cdots)\\&\mu\left(\frac{\pi h}{a}\right)^2\left(\frac{m}{n}\right)^2\left[\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2\right]A_n+4\mu\left(\frac{m}{na}\right)^2Dh\\&=-\frac{2\sigma}{n\pi}\sin\frac{n\pi}{2}\qquad(n=1,3,\cdots)\end{aligned} \tag{5.8}$$

In view of (2.4e) and (5.6), we have

$$2\mu\left(\frac{m\pi}{a}\right)^2D\frac{h^3}{12}+2\mu\left(\frac{h}{a}\right)^2\sum_{n=1,3\cdots}^{\infty}\frac{\left(\frac{n}{h}\right)^2+2\left(\frac{m}{a}\right)^2}{n^2}m\sqrt{\left(\frac{n}{h}\right)^2+\left(\frac{m}{a}\right)^2}\left(\frac{h}{n}\right)^2=H_0 \tag{5.9}$$

Equations (5.8), together with equations (5.9), yield

$$D=\frac{s}{\mu h}$$

$$E=-\frac{s}{2\mu}$$

$$A_n=\begin{cases} -\dfrac{2}{\mu}\left(\dfrac{na}{m\pi h}\right)^2\cdot\dfrac{\dfrac{\sigma}{n\pi}\sin\dfrac{n\pi}{2}+2\left(\dfrac{m}{na}\right)^2 s}{\left(\dfrac{n}{h}\right)^2+\left(\dfrac{m}{a}\right)^2} & (n=1,\ 3,\ 5,\ \cdots) \\ 0 \quad (n=2,\ 4,\ 6,\ \cdots) & \end{cases}$$

Here

$$s=\frac{6H_0+\dfrac{24\sigma h^2 a}{\pi^3 m}\displaystyle\sum_{l=1,3,5,\cdots}^{\infty}\frac{\left(\frac{l}{h}\right)^2+2\left(\frac{m}{a}\right)^2}{l^3\sqrt{\left(\frac{l}{h}\right)^2+\left(\frac{m}{a}\right)^2}}\sin\frac{l\pi}{2}}{\left(\dfrac{m\pi h}{a}\right)^2-\dfrac{48m}{\pi^2 a}\displaystyle\sum_{l=1,3,\cdots}^{\infty}\frac{\left(\frac{l}{h}\right)^2+2\left(\frac{m}{a}\right)^2}{l^4\sqrt{\left(\frac{l}{h}\right)^2\left(\frac{m}{a}\right)^2}}} \tag{5.10}$$

If the sum of the first two terms is used to replace the sum of the infinite series, the error will be less than 5% in usual case.

The infinite series in (5.7a) and (5.7b) converge rapidly, since the general terms decay according to exponential law as the value of n increases. Hence, the sum of the infinite series can be replaced with the sum of the first two terms in most cases.

In the case of plane deformation, one component among u, v, w is zero, so the expressions for displacements are much more simple. For instance, if $w=0$, we can put $D=E=0$ in (5.7). Relieve the confinement by (5.6) and remain the boundary condition equation(5.5), the results thus become

$$u(x,y,z)=-\frac{2\sigma}{\mu\pi^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l-1}\cdot\frac{1}{\sqrt{\left(\frac{2l-1}{h}\right)^2+\left(\frac{m}{2}\right)^2}}\cos\frac{(2l-1)\pi z}{h}$$
$$\exp\left[-\pi\sqrt{\left(\frac{2l-1}{h}\right)^2+\left(\frac{m}{a}\right)^2}\,x\right]\sin\frac{m\pi y}{a}$$

$$v(x,y,z)=\frac{2\sigma a}{\mu\pi^2 m}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l-1}\cos\frac{(2l-1)\pi z}{h}$$
$$\exp\left[-\pi\sqrt{\left(\frac{2l-1}{h}\right)^2+\left(\frac{m}{a}\right)^2}\,x\right]\cos\frac{m\pi y}{a}$$

$$w(x,\ y,\ z)=0$$

2. Simply supported rectangular plate with plate-surface loads

Here takes the case where the upper surface of the plate is acted with a concentrated force. The other else cases with arbitrary distributed loads on the plate-surface can be easily formulated by using the results provided here.

Suppose that a concentrated force P is acted on the upper surface of $a\times b$ rectangular plate. It follows from Ref. [9] that

$$q_{mn}=\frac{4P}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} \tag{5.11}$$

The expressions for the displacements can be obtained immediately upon substituting of (5.11) into (4.6) and (4.7).

In the case of the thin plate, owing to the thickness is small, the results of the classical thin plate theory approach to those of the present paper's. In fact, put $h\to 0$ in (4.6c), then

$$\varphi_{mn}(z)\sim\frac{3(\lambda+2\mu)q_{mn}}{\mu(\lambda+\mu)\pi^4\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2h^3} \tag{5.12}$$

The flexural rigidity of the plate is

$$D=\frac{Eh^3}{12(1-\nu^2)} \tag{5.13}$$

where E is Young's modules and ν is Poisson's ratio. They are related to lamé's constants as follows:

$$\mu=\frac{E}{2(1+\nu)},\quad \frac{\lambda+2\mu}{\lambda+\mu}=2(1-\nu) \tag{5.14}$$

Substituting (5.13), (5.14) and (5.11) into (5.12) leads to

$$\varphi_{mn}(z)\sim\frac{4P\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{abD\pi^4\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}$$

Substituting the above expression into (4.7c) gives the deflection at the point (x, y) of the plate, on whose surface a concentrated force P is acted at the point $(x=\xi,\ y=\eta)$, as

$$w(x,y,z)\sim\frac{4P}{abD\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$
$$\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

The maximum deflection w_{max} occurs at the center $\left(x=\frac{a}{2},\ y=\frac{b}{2}\right)$ of the plate. When the concentrated force P is also located at the center $\left(\xi=\frac{a}{2},\ \eta=\frac{b}{2}\right)$ of the plate,we have

$$w_{max}=w\Big|_{x=\frac{a}{2},y=\frac{b}{2}}\sim\frac{4P}{abD\pi^4}\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}\frac{1}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2} \tag{5.15}$$

The same results were obtained by Ref. [9], which solved the same problem with Navier method. From this fact it can be concluded that the solution of Navier method is the limit case of the present paper's result as $h\to 0$. This makes it clear that the classical plate theory based on Kirchhoff-Love hypothesis is reasonable when the plate is thin.

VI. Discussion and Conclusion

The precise solutions of closed form, which suits to the plates of any thickness, are obtained by proceeding from Navier equations of three dimensional elastic body.

The solutions obtained in this paper satisfy exactly the governing equations, and can reflect the thickwise distribution of sresses or displacements on the edges of plates. This is the main difference between the method of the present paper and that of the other else plate theories.

The expressions (3.8) can suit to the plates acted with complicatedly distributed stresses on the edges. For a given distributing law of the edge stresses, the distributing function of the edge-stresses can be expanded into trigonometric series to find the solutions.

The solution constituted from (4.5) and (4.6) can be used to analyze the problems of arbitrary distributed plate-surface loads. The method is here again by expanding the distributing function of the plate-surface loads into trigonometric series.

The case contains both arbitrary distributed plate-surface loads and edge stresses can also be analyzed by using the superposition of the solutions constituted from (3.8) and those constituted from (4.5) and (4.6).

When the plate is very thin and it has simultaneously the bending deflection and the inplane compression, the coupling effects between the two types of deformations are intensive. Then, it is unallowable to superpose the in-plane deformation due to the in-plane compression and the bending deflection. Otherwise, large error will be caused.

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