Applied Mathematics and Mechanics
(English Edition, Vol. 18, No. 9, Sep. 1997)

Published by SU, Shanghai, China

A NEW ALGORITHM FOR SOLVING DIFFERENTIAL/ALGEBRAIC EQUATIONS OF MULTIBODY SYSTEM DYNAMICS*

Wang Yibing (王艺兵)[1] Zhao Weijia (赵维加)[1] Pan Zhenkuan (潘振宽)[1]

(Received Dec. 18, 1996; Communicated by Cheng Gengdong)

Abstract

The second order Euler-Lagrange equations are transformed to a set of first order differential/algebraic equations, which are then transformed to state equations by using local parameterization. The corresponding discretization method is presented, and the results can be used to implementation of various numerical integration methods. A numerical example is presented finally.

Key words multibody systems, differential/algebraic equations, numerical analysis

I. Introduction

The equations of motion of constrained multibody system dynamics can be written as follows[1]

$$\left.\begin{aligned} &M(q,t)\ddot{q}+\phi_q^T(q,t)\lambda=F(q,\dot{q},t) && (1.1a)\\ &\phi(q,t)=0 && (1.1b)\end{aligned}\right\}$$

where t is time parameter, $q\in R^n$ is generalized coordinates vector of system, $M(q,t)$: $R^n\times R\to R^{n\times n}$ is generalized mass matrix of system, $\lambda\in R^m$ is Lagrange multiplier vector, $\phi(q,t)$: $R^n\times R\to R^m(m<n)$ is kinematic constraint function vector; $\phi_q(q,t)$ is derivative of ϕ with respect to q (Jacobian matrix); $F(q,\dot{q},t)$: $R^n\times R^n\times R\to R^n$ is generalized force vector. Eq. (1.1) is called the second class mathematic model of multibody system dynamics[10] or Euler-Lagrange equations[9]. Many numerical approaches for this model are compared in [13].

In this paper, t is augmented into the vector of coordinates, and the second order model is transformed to first order model with index one firstly. Then the concept of local equivalent model is suggested, based on which a new discretization technique is presented. The validity of the approach is tested through a numerical example finally.

II. Transformation of Second Order Euler-Lagrange Equations to First Order Ones

Similar to reference [2], (1.1a), (1.1b) satisfy the following conditions:

(1) ϕ, M, F are smooth sufficiently;

(2) M is a positive and symmetric matrix;

(3) The row rank of ϕ_q is full.

* Project supported by Natural Science Foundation of China and Shangdong Province

[1] Department of Automation, Qingdao University, Qingdao 266071, P. R. China

In order to simplify the structure of equations, the following matrices are defined based on (1.1a) and (1.1b)

$$q_1=\begin{bmatrix} q \\ t \end{bmatrix} \quad F_1=\begin{bmatrix} F \\ 0 \end{bmatrix} \quad M_1=\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$$

Then (1.1a) and (1.1b) can be transformed to

$$\left.\begin{aligned} &M_1(q_1)\ddot{q}_1+\phi_{q_1}^T(q_1)\lambda_1=F_1(q_1,\dot{q}_1) && (2.1a)\\ &\phi(q_1)=0 && (2.1b)\end{aligned}\right\}$$

Obviously, M_1 is positive, and the row rank of $\phi_{q_1}=[\phi_q,\phi_t]$ is full, i. e., the assumptions of (1.1a) and (1.1b) can be satisfied. Here, $\phi_q=\partial\phi/\partial q$, $\phi_t=\partial\phi/\partial t$, λ_1 is a new Lagrange multiplier vector. In order to write in compact form, (2.1a) and (2.1b) are rewritten as follow

$$\left.\begin{aligned} &M(q)\ddot{q}+\phi_q^T(q)\lambda=F(q,\dot{q}) && (2.2a)\\ &\phi(q)=0 && (2.2b)\end{aligned}\right\}$$

The following two equations can be deduced from (2.2b)

$$\phi_q\dot{q}=0 \tag{2.2c}$$

$$\phi_q\ddot{q}=-\phi_{qq}(\dot{q}\cdot\dot{q}) \tag{2.2d}$$

The following equations can be obtained through combination of (2.2a) and (2.2d)

$$\begin{bmatrix} M & \phi_q^T \\ \phi_q & 0 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix}=\begin{bmatrix} F \\ -\phi_{qq}(\dot{q}\cdot\dot{q}) \end{bmatrix} \tag{2.3}$$

Because M is positive and the row rank of ϕ_q is full, the equation (2.3) have unique solutions as follow

$$\ddot{q}=\varphi(q,\dot{q}) \tag{2.4}$$

where,

$$\varphi(q,\dot{q})=[I,0]\begin{bmatrix} M & \phi_q^T \\ \phi_q & 0 \end{bmatrix}^{-1}\begin{bmatrix} F \\ -\phi_{qq}(\dot{q}\cdot\dot{q}) \end{bmatrix} \tag{2.5}$$

After introducing new coordinates vector $x=[q^T,\dot{q}^T]^T$, (2.4) becomes

$$\dot{x}=g(x) \tag{2.6}$$

where,

$$g(x)=\begin{bmatrix} \dot{q} \\ \varphi(q,\dot{q}) \end{bmatrix}$$

(2.2b) and (2.2c) can be changed to

$$f(x)=0 \tag{2.7}$$

here,

$$f(x)=\begin{bmatrix} \phi(q) \\ \phi_q(q)\dot{q} \end{bmatrix}$$

(2.6) and (2.7) are first order equivalent equations of (2.2a) and (2.2d) with index one.

III. Local Equivalent Equations and Their Discretization

If $x=x(t)$ is assumed a set of solutions of (2.6) and (2.7), the following equations can be obtained by substituting x into (2.7):

$$Df(x)\dot{x}=Df(x)g(x)=0 \tag{3.1}$$

(2.8) is implicit condition of (2.6) and (2.7), and can be considered as consistent condition of (2.6) and (2.7). The following overdetermined equations can be obtained through combination of (2.6), (2.7) and (3.1).

$$\dot{x}=g(x) \tag{3.2a}$$

$$f(x)=0 \tag{3.2b}$$

$$Df(x)g(x)=0 \tag{3.2c}$$

Their initial conditions are

$$x(t_0)=x_0 \tag{3.3}$$

which is a typical initial value problem of first order differential/algebraic equations. Here, x_0 is called consistent initial value if it satisfies (3.2b) and (3.2c).

Proposition 1 $f(x)$, $g(x)$ are same as (2.6) and (2.7). Let A be a $2n\times 2p$ matrix, where, $p=n-m$. If A is selected such that

$$\begin{bmatrix} A^T \\ Df(x_0) \end{bmatrix}$$

is nonsingular, there exists a small adjacent field δ_{t_0} of t_0, on which (3.2a ~ 3.2c) have same solutions with the following equations under the same initial conditions (3.3).

$$A^T(\dot{x}-g(x))=0 \tag{3.4a}$$

$$f(x)=0 \tag{3.4b}$$

$$Df(x)g(x)=0 \tag{3.4c}$$

Proof Only if obtaining (3.2a) from (3.4a) and (3.4c). It can be assumed that $x=x(t)$ is a set of solutions of (3.4a~3.4c) and (3.3). Because $Df(x)$ is continuous, there exists a small adjacent field δ_{t_0} of t_0, on which

$$\begin{bmatrix} A^T \\ Df(x) \end{bmatrix}$$

is nonsingular, then the following equation can be obtained from (3.4b)

$$Df(x)\dot{x}=0 \tag{3.5}$$

Thus, the following deductions can be made from (3.4a), (3.4c) and (3.5)

$$\dot{x}=\begin{bmatrix} A^T \\ Df(x) \end{bmatrix}^{-1}\begin{bmatrix} A^T g(x) \\ 0 \end{bmatrix}=\begin{bmatrix} A^T \\ Df(x) \end{bmatrix}^{-1}\begin{bmatrix} A^T g(x) \\ Df(x)g(x) \end{bmatrix}$$

$$=\begin{bmatrix} A^T \\ Df(x) \end{bmatrix}^{-1}\begin{bmatrix} A^T \\ Df(x) \end{bmatrix}g(x)=g(x)$$

End.

Note For common cases, if there is no (3.4c), the equations (3.4a) and (3.4b) are not equivalent with (3.2a) and (3.2b). For example, the following equations

$$\left.\begin{array}{l}\dot{x}=x+t\\ \dot{t}=1\\ x+t=1\end{array}\right\}$$

are conflict equations because of $Df(x)g(x)=x+t+1=2\neq 0$. But if A^T is [0, 1], (3.4a) and (3.4b) are changed to

$$\dot{t}=1$$
$$x+t=1$$

They have solutions under consistent initial values.

Proposition 2 If $f(x)$ and $g(x)$ are difined as (2.6) and (2.7), solutions of (3.4a) and (3.4b) satisfy the consistent initial condition (3.4c).

Proof From (2.6) and (2.7), the following deductions

$$Df(x)g(x)=\begin{bmatrix}\phi_q & 0\\ \phi_{qq}\dot{q} & \phi_q\end{bmatrix}\begin{bmatrix}\dot{q}\\ \varphi(q,\dot{q})\end{bmatrix}=\begin{bmatrix}\phi_q\dot{q}\\ \phi_{qq}(\dot{q}\cdot\dot{q})+\phi_q\varphi(q,\dot{q})\end{bmatrix} \tag{3.6}$$

are correct. The upper part of right side of (3.6) is 0 permanently from the definition of $\varphi(q,\dot{q})$ and the corresponding lower part is included in (3.4b). End.

(3.4a) and (3.4b) can be discretized to a set of common equations because they are not overdetermined equations. The modified Euler method will be used to examplify the above ideas. The marix A depends on initial values of every step associated with difference grids of (3.4a), (3.4b).

If $x_0=[q_0^T,\dot{q}_0^T]^T$, for $k=0,1,2,\cdots,x_{k+1}$ can be obtained from the following equations

$$\begin{cases}A_k^T(x-x_k-h/2(g(x)+g(x_k)))=0 & (3.7\text{a})\\ f(x)=0 & (3.7\text{b})\end{cases}$$

They are nonlinear algebraic equations with $2n$ variables. If Newton iteration method is used to (3.7b), and direct iteration method is used to (3.7a), the following computing technique can be obtained

$$A_k^T\delta_s=-A_k^T(x_s'-x_k-h/2(g(x_s')+g(x_k))) \tag{3.8a}$$
$$Df(x_s')\delta_s=-f(x_s') \tag{3.8b}$$
$$x_{s+1}'=x_s'+\delta_s \quad s=0,1,2,\cdots,l;\ x_0'=x_k;\ x_l'=x_{k+1};k=0,1,2;\ \cdots$$

IV. Selection of A_k and Steps of Computation

Because the row rank of ϕ_q is full, the solution space of $\phi_q u=0$ is $p=n-m$ dimension, whose base vectors can be assumed to be $V=[V_1,V_2,\cdots,V_p]$. The following equation can obtained by left multiplying (2.2a) with V^T

$$V^T M\ddot{q}=V^T F \tag{4.1}$$

(4.1) and (2.2d) can be combined to:

$$\begin{bmatrix}V^T M\\ \phi_q\end{bmatrix}\ddot{q}=\begin{bmatrix}V^T F\\ -\phi_{qq}(\dot{q}\cdot\dot{q})\end{bmatrix} \tag{4.2}$$

Proposition 3 The coefficient matrix of (4.2)

$$\begin{bmatrix} V^T M \\ \phi_q \end{bmatrix} \tag{4.3}$$

is nonsingular.

Proof Let $u \in R^n$ satisfy

$$\begin{bmatrix} V^T M \\ \phi_q \end{bmatrix} u = \begin{bmatrix} V^T M u \\ \phi_q u \end{bmatrix} = 0$$

then u is located in the null space of ϕ_q, i. e., there exists $y \in R^q$ such that $u = Vy$, thus

$$V^T M V y = 0$$

i. e.,

$$(Vy)^T M V y = 0$$

Because M is positive and symmetric, $u = Vy = 0$ can be obtained. End.

Note Because M is positive and symmetric, $M^T V$ is orthogonized to $\phi_q{}^T$ under the sense of $\|.\|_{M^{-1}}$[11]. Thus, $B = V^T M$ is one of selections such that the condition number of

$$\begin{bmatrix} B \\ \phi_q \end{bmatrix}$$

is smaller.

Deduction 1 (2.3) and (4.2) are sets of equivalent differential equations.

Deduction 1 demonstrates that $\varphi(q, \dot q)$ in (2.4) can be obtained from (4.2). The matrix A_k in (3.7a~3.7b) can be selected according to proposition 3 and

$$Df(x) = \begin{bmatrix} \phi_q & 0 \\ \phi_{qq}\dot q & \phi_q \end{bmatrix}$$

i. e.,

$$A_k = \begin{bmatrix} V^T M & 0 \\ \phi_q & V^T M \end{bmatrix}_{x = x_k} \tag{4.4}$$

where $x_k = [q_k^T, \dot q_k^T]^T$, $(A)_{x=x_k}$ is value of A at point x_k. Let $\delta_s = [\delta_{1s}^T, \delta_{2s}^T)^T$ and substitute (4.4) into (3.8a~3.8b), then (3.8a) and (3.8b) become

$$\begin{bmatrix} V^T M & 0 \\ 0 & V^T M \\ \phi_q & 0 \\ \phi_{qq}\dot q & \phi_q \end{bmatrix}_s \begin{bmatrix} \delta_{1s} \\ \delta_{2s} \end{bmatrix} = \begin{bmatrix} -V^T M(x_s - x_k - h/2(g(x_s) + g(x_k)))_1 \\ -V^T M(x_s - x_k - h/2(g(x_s) + g(x_k)))_2 \\ (-f(x_s))_1 \\ (-f(x_s))_2 \end{bmatrix} \tag{4.5}$$

where, $(x)_1$ is a vector composed of first n components of x, and $(x)_2$ is another vector composed of last n components of x. Due to $(x)_1 = q$, $(x)_2 = \dot q$, $(g(x))_1 = \dot q$, $(g(x))_2 = \varphi(q, \dot q)$, $(f(x))_1 = \phi(q)$, $(f(x))_2 = \phi_q \dot q$, the equations 1, 3 and 2, 4 of (4.5) can combined to the following equations

$$\begin{bmatrix} V^T M \\ \phi_q \end{bmatrix}_s \delta_{1s} = \begin{bmatrix} -V^T M(q_s' - q_k - h/2(\dot q_s' - \dot q_k)) \\ -\phi(q_s') \end{bmatrix} \tag{4.6}$$

$$\begin{bmatrix} V^T M \\ \phi_q \end{bmatrix}_s \delta_{2s} = \begin{bmatrix} -V^T M(\dot{q}_s' - \dot{q}_k - h/2(\ddot{q}_s' - \ddot{q}_k)) \\ -(\phi_{qq}\dot{q}' \cdot \delta_{1s} - (\phi_q \dot{q}')_s \end{bmatrix} \tag{4.7}$$

$$q_{s+1}' = q_s' + \delta_{1s}, \quad \dot{q}_{s+1}' = \dot{q}_s' + \delta_{2s} \qquad (s=0,1,2,\cdots), \quad q_0' = q_k, \quad \dot{q}_0' = \dot{q}_k.$$

The computation steps can be described as follows

For the kth step: q_k, $\dot{q}_k$ are known. let $q_0' = q_k$, $\dot{q}_0' = \dot{q}_k$. For $s=0,1,2,\cdots$

(1) Calculation of $M, F, \phi, \phi_q, \phi_q \dot{q}, \phi_{qq}(\dot{q}\cdot\dot{q})$, at $q_s, \dot{q}_s$,

(2) Calculation of base vectors V from $\phi_q x = 0$.

(3) Obtaining $\ddot{q}_s'$ from (4.2),

(4) Obtaining q_{s+1}', $\dot{q}_{s+1}'$ from (4.6), (4.7),

(5) If $\|q_{s+1}' - q_s'\| < \varepsilon$ and $\|\dot{q}_{s+1}' - \dot{q}_s'\| < \varepsilon$, then let $q_{k+1} = q_{s+1}'$, $\dot{q}_{k+1} = \dot{q}_{s+1}'$ and go to the $(k+1)th$ step.

V. Selection of Matrix V

5.1 Orthogonization method

The QR orthogonal decomposition[11] of ϕ_q^T can be written as

$$\phi_q^T = [U, \ V] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

where, $[U, V]$ is orthogonal square matrix, R is upper triangle matrix, V have P orthogonal columns. The matrix V can obtained from the following equation

$$\phi_q V = [R^T, 0] \begin{bmatrix} U^T \\ V^T \end{bmatrix} V = [R^T, 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$$

5.2 Gauss elimination method

Because the row rank of ϕ_q is full, the following results can be obtained through row pivot Gauss elimination of ϕ_q, i. e.,

$$P\phi_q Q = \begin{bmatrix} e_{p+1}{}^T \\ \vdots \\ e_n{}^T \end{bmatrix} \qquad e_i = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]$$

where, P is elimination matrix, Q is multiplication of element column transformation matrices. V can be selected as

$$V = Q[e_1, \cdots, e_p]$$

thus

$$P\phi_q Q = \begin{bmatrix} e_{p+1}{}^T \\ \vdots \\ e_n{}^T \end{bmatrix} [e_1, \cdots, e_p] = 0$$

$\phi_q V = 0$, because P is nonsingular.

The Gauss elimination method can maintain sparsity of matrices and have simple computation structure. The orthogonization method have an advantage of small condition number of matrices.

VI. Numerical Example

A planar manipulator with two links is shown in Fig. 1, the lengths and masses of links are $l_1=1\text{m}$, $l_2=2\text{m}$, $m_1=1\text{kg}$, $m_2=1\text{kg}$, respectively. The vector of coordinates is

$$q=[x_1 \quad y_1 \quad \theta_1 \quad x_2 \quad y_2 \quad \theta_2]^T$$

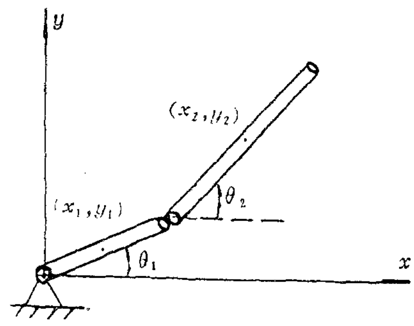

Fig. 1 Two link manipulator

The generalized mass matrix of the system is:

$$M=\begin{bmatrix} m_1 & & & & & \\ & m_1 & & & & \\ & & J_1 & & & \\ & & & m_2 & & \\ & & & & m_2 & \\ & & & & & J_2 \end{bmatrix}$$

The vector of generalized forces is:

$$F=[0 \quad -m_1 g \quad 0 \quad 0 \quad -m_2 g \quad 0]^T$$

The constraint equations are:

$$x_1-(l_1/2)\cos\theta_1=0$$
$$y_1-(l_1/2)\sin\theta_1=0$$
$$x_2-l_1\cos\theta_1-(l_2/2)\cos\theta_2=0$$
$$y_2-l_1\sin\theta_1-(l_2/2)\sin\theta_2=0$$

The initial conditions are:

$$q_0=[0.5 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0]^T$$
$$\dot{q}_0=[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

Because the analytical results are impossible to obtain, the approximate numerical results from QR method[4] with $h=0.001\text{s}$ are regarded as nearly exact values used to compare with those obtained through the method presented in this paper. The Fig. 2 is a result from QR method, and the Fig. 3 is deviation of y_1 from the values shown in Fig. 2, through the new method with $h=10^{-2}\text{s}$, error tolerance EPS$=10^{-4}$.

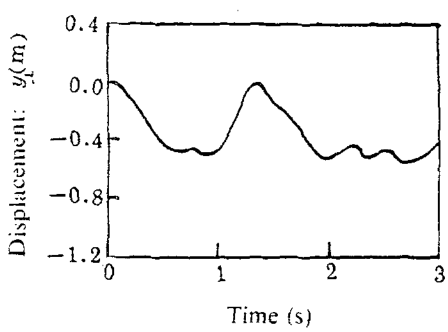

Fig. 2 The history of y_1

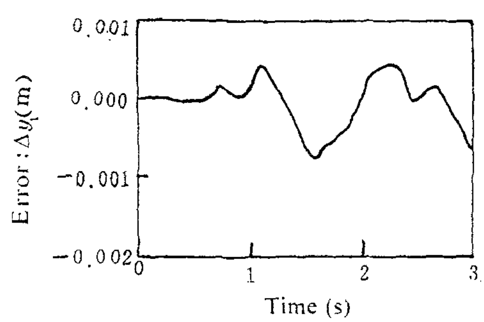

Fig. 3 The deviation of y_1 from the values in Fig. 2

VII. Conclusions

The second order Euler-Lagrange equations are transformed to a set of first order differential/algebraic equations, which are then transformed to state equations by using local parameterization. The corresponding discretization method is presented. Some features of this algorithm are:

a. t is augmented into q to simplify the structures of equations;

b. In the main iteration steps (2.2) and (2.3), the equations (4.2), (4.6) and (4.7) have same coefficient matrices, therefore, they can be calculated one time at evey computation step;

c. Because $V^T M$ is orthogonized to $\phi_q A^{-1}$, when V is an approximate orthogonal matrx, the stiffness of equations depend on ϕ_q only;

d. The method presented in this paper can be implemented by various discretization technique, such as Runge-Kutta method.

References

[1] E. J. Haug, *Computer Aided Kinematics and Dynamics of Mechanical Systems*, Vol. 1: Basic Methods, Ally & Bacon, Boston, MA (1989).

[2] F. A. Potra and W. C. Rheinbolt, On the numerical solution of Euler-Lagrange equations, *Mechanics of Structures & Machines*, **19**, 1 (1991), 1~18.

[3] J. W. Baumgarte, A new method of stabilization for holonomic constraints, *J. Applied Mechanics*, **50** (1983), 869~870.

[4] R. P. Singh and P. W. Likins, Singular value decomposition for constrained dynamical systems, *J. Applied Mechanics*, **52** (1985), 943~948.

[5] S. S. Kim and M. J. Vanderploeg. QR decomposition for state space representation of constrained mechanical dynamic systems, *J. Mech. Tran. & Auto. in Design*, **108** (1986), 168~183.

[6] C. G. Liang and G. M. Lance, A differential null space method for constrained dynamic analysis, *J. Mech. Tran. & Auto. in Design*, **109** (1987), 405~411.

[7] O. P. Agrawal and S. Saigal, Dynamic analysis of multibody systems using tangent coordinates, *Computers & Structures*, **31**, 3 (1989), 349~355.

[8] J. W. Kamman and R. L. Huston, Constrained multibody systems—An automated approach, *Computers & Structures*, **18**, 4 (1984), 999~1112.

[9] F. A. Potra and J. Yen, Implicit integration for Euler-Lagrange equations via tangent space parameterization, *Mechanics of Structures & Machines*, **19**, 1 (1991), 77~98.

[10] Hong Jiazhen and Liu Yanzhu, Computational dynamics of multibody systems, *Advancement of Mechanics*, **19**, 2 (1989), 205~210. (in Chinese).

[11] Wang Deren, *Numerical Methods for Nonlinear Equations and Optimization Techniques*, People's Education Press, Beijing (1980). (in Chinese)

[12] Pan Zhenkuan, The modelling theory and numerical study for dynamics of flexible multibody systems, Doctoral dissertation, Shanghai Jiao tong University (1992). (in Chinese)

[13] Pan Zhenkuan, Zhao Weija, Hong Jiazhen and Liu Yanzhu, On numerical techniques for differential/algebraic equations of motion of multibody system dynamics, *Advancement of Mechanics*, **26**, 1 (1996), 28~40. (in Chinese)