

## SPLITTING METHOD FOR TWO-DIMENSIONAL PHREATIC FLOW EQUATION

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### Abstract

*In this paper, according to their difference in "physical" meaning, two-dimensional phreatic flow equation, which has been transformed, is divided into two parts—advection and dispersion by the splitting method. For the former, alternating direction finite difference method will be used and for the latter, it is resolved by alternating direction Picard iteration, therefore, the aim of computing the solution of whole problem will be reached. At last, the validity of the algorithms is proved by the numerical example. The comparison of the proposed method with the conventional finite difference (linearized) is made. The results show that the precision of calculation by the method proposed in this paper is better than the conventional methods.*

**Key words** phreatic flow equation, nonlinear, splitting method, finite difference method, Picard iteration

### I. Introduction

As well-known, that the governing equation of two-dimensional phreatic flow

$$-\frac{\partial}{\partial x} \left( \kappa H \frac{\partial H}{\partial x} \right) + \frac{\partial}{\partial y} \left( \kappa H \frac{\partial H}{\partial y} \right) + W = \mu \frac{\partial H}{\partial t}$$

is a nonlinear parabolic equation. At present, it is a usual approach to replace  $\kappa H$  in bracket with  $\kappa \bar{H}$  ( $\bar{H}$  is the average thickness) in the research of groundwater flow modeling, then,  $\kappa \bar{H}$  may be considered as constant and moved outside of bracket. The equation mentioned above will be changed into

$$\kappa \bar{H} \frac{\partial^2 H}{\partial x^2} + \kappa \bar{H} \frac{\partial^2 H}{\partial y^2} + W = \mu \frac{\partial H}{\partial t}$$

it is similar to the governing equation of confined water flow (a linear parabolic equation) which can be solved by the conventional methods such as finite difference method, finite element method and boundary element method.

The author holds, that these techniques are feasible under the condition of not larger changing scope  $\Delta H$  of phreatic water head  $H$ , but, the larger error will be caused if these methods are still used when the  $\Delta H$  are larger. For ensuring the accuracy of problem to be solved, in this paper, on the basis of transforming the phreatic flow equation, the goal of numerical computation will be achieved by the splitting method.

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## II. The Transformation of Two-Dimensional Phreatic Flow Equation

Let's consider a simple case, provided that phreatic aquifer is homogeneous, isotropic, horizontal bottom and no source (or sink) terms, then governing equation describing such a geological model can be expressed as

$$-\frac{\partial}{\partial x}\left(\kappa H \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa H \frac{\partial H}{\partial y}\right) = \mu \frac{\partial H}{\partial t} \quad ((x, y) \in \Omega, t \in (0, T]) \quad (2.1)$$

where  $H(x, y, t)$ —phreatic head (L),  $\kappa$ —hydraulic conductivity ( $LT^{-1}$ ),  $\mu$ —specific yield,  $t$ —time (T),  $x, y$ —space coordinates (L).

initial condition:

$$H(x, y, 0) = H_0(x, y), \quad (x, y) \in \Omega$$

boundary conditions:

$$\begin{aligned} H(x, y, t)|_{\Gamma_1} &= H_1(x, y, t), \quad (x, y) \in \Gamma_1 \\ K \frac{\partial H(x, y, t)}{\partial \eta} \Big|_{\Gamma_2} &= q(x, y, t), \quad (x, y) \in \Gamma_2 \end{aligned}$$

The first term on the left side in Eq. (2.1)

$$-\frac{\partial}{\partial x}\left(\kappa H \frac{\partial H}{\partial x}\right) = \kappa \frac{\partial H}{\partial x} \frac{\partial H}{\partial x} + \kappa H \frac{\partial^2 H}{\partial x^2} \quad (2.2)$$

The second term on the left side in Eq. (2.1)

$$\frac{\partial}{\partial y}\left(\kappa H \frac{\partial H}{\partial y}\right) = \kappa \frac{\partial H}{\partial y} \frac{\partial H}{\partial y} + \kappa H \frac{\partial^2 H}{\partial y^2} \quad (2.3)$$

Substitute (2.2) and (2.3) into (2.1):

$$\kappa \frac{\partial H}{\partial x} \frac{\partial H}{\partial x} + \kappa H \frac{\partial^2 H}{\partial x^2} + \kappa \frac{\partial H}{\partial y} \frac{\partial H}{\partial y} + \kappa H \frac{\partial^2 H}{\partial y^2} = \mu \frac{\partial H}{\partial t} \quad (2.4)$$

here  $\kappa \frac{\partial H}{\partial x} = v_x$ ,  $\kappa \frac{\partial H}{\partial y} = v_y$ ,  $v_x$ ,  $v_y$  is seepage velocity in the  $x, y$ , direction, respectively, so

Eq. (2.4) can be written as:

$$v_x \frac{\partial H}{\partial x} + v_y \frac{\partial H}{\partial y} + \kappa H \frac{\partial^2 H}{\partial x^2} + \kappa H \frac{\partial^2 H}{\partial y^2} = \mu \frac{\partial H}{\partial t} \quad (2.5)$$

## III. The Splitting of the Transformed Phreatic Flow Equation

Eq. (2.5) is similar to a advection-dispersion equation, the first two terms of left side is corresponding to "advection" part, the last two terms can be considered as "dispersion" part. According to their difference in "physical" meanings, Eq. (2.5) may be factorized into (3.1) and (3.2) in time interval  $(n\Delta t, (n+1)\Delta t)$ :

$$v_x \frac{\partial H_1}{\partial x} + v_y \frac{\partial H_1}{\partial y} = \frac{1}{2} \mu \frac{\partial H_1}{\partial x} \quad (3.1)$$

$(x, y) \in \Omega$ ,  $t \in (n\Delta t, (n + \frac{1}{2})\Delta t)$ ,  $\Omega$  is the computational region

The initial condition is:  $H_1(x, y, n\Delta t) = H(x, y, n\Delta t)$

The boundary conditions are equivalent to the unsplitting ones.

$$\kappa H_2 \frac{\partial^2 H_2}{\partial x^2} + \kappa H_2 \frac{\partial^2 H_2}{\partial y^2} = \frac{1}{2} \mu \frac{\partial H_2}{\partial t} \quad (3.2)$$

$$(x, y) \in \Omega, \quad t \in \left( \left( n + \frac{1}{2} \right) \Delta t, (n+1) \Delta t \right)$$

the initial conditions is:  $H_2\left(x, y, \left(n + \frac{1}{2}\right) \Delta t\right) = H_1\left(x, y, \left(n + \frac{1}{2}\right) \Delta t\right)$

all the same, its boundary conditions is also equal to the unfactorized ones.

where  $H_1(x, y, t)$  = "advectional" head (L)

$H_2(x, y, t)$  = "dispersional" head (L)

$\Delta t$  = time interval (T)

$T$  = prescribed computational time (T)

Putting the two problems above together. We have:  $H(x, y, (n+1) \Delta t) = H_2(x, y, (n+1) \Delta t)$ , that is to say, the hydraulic heads at the  $n\Delta t$  moment are used as the initial values of hydraulic heads  $H_1$  of "advectional" problem and  $H_1$  is computed within the first half time step  $(1/2) \Delta t$ . The values of  $H_1(x, y, t)$  at the  $(n+1/2) \Delta t$  moment act as the initial values of the "dispersional" problem in the last half time step  $1/2 \Delta t$ . Furthermore, we can resolve the hydraulic head values  $H_2$  of "dispersional" problem at the  $(n+1) \Delta t$  moment,  $H_2(x, y, (n+1) \Delta t)$ , is the results of whole issue at the  $(n+1) \Delta t$  moment.

If  $v_x, v_y$  have been obtained by using the hydraulic heads at the last moment, Eq. (3.1) will be a two-dimensional hyperbolic equation Eq. (3.2) is still a nonlinear parabolic one. For the convenience of expression, when the algorithms of Eq. (3.1) and Eq. (3.2) are derived, the subscripts "1" and "2" about  $H_1(x, y, t)$  and  $H_2(x, y, t)$  will be omitted in the following parts.

#### IV. Finite Difference Method for the Factorized Equation

Firstly, let's derive the finite difference method of (3.1). In order to keep the smoothness of  $v_x, v_y$ , and enhance their accuracy, we would compute  $\partial H / \partial x$  and  $\partial H / \partial y$  at the last moment, by employing cubic spline interpolation on the difference grid.

Assumed that the whole computational domain has been discretized into rectangular grids, there are  $N+1$  nodes in the  $x$  direction,  $M+1$  nodes in the  $y$  direction. In the range  $(x_i, x_{i+1})$  of  $x$  direction, provided that for cubic spline interpolation polynomial  $s_1(x_i, y_j)^{[1]}$ , the interpolational conditions are satisfied.  $s_1(x_i, y_j) = H_{i,j} (i=1, 2, \dots, N)$  and  $s_1(x_i, y_j)$  as well as its first, second partial derivatives at inner nodes  $(x_i, y_j) (i=1, 2, \dots, N-1)$  are continuous.

Let the first derivatives of  $s_1(x_i, y_j)$  ( $i=1, 2, \dots, N$ ) be  $m_{i,j}$  ( $i=1, 2, \dots, N$ ), then we may derive the following equation group:

$$\lambda_i m_{i+1,j} + 2m_{i,j} + \mu_{i,j} m_{i+1,j} = g_{i,j} \quad (i=1, 2, \dots, N-1) \quad (4.1)$$

where

$$\lambda_i = \frac{h_i}{h_{i-1} + h_i}, \quad \mu_i = 1 - \lambda_i \quad (i=1, 2, \dots, N)$$

$$g_{i,j} = 3 \left( \lambda_i \frac{H_{i,j} - H_{i-1,j}}{h_{i-1}} + \mu_i \frac{H_{i+1,j} - H_{i,j}}{h_i} \right) \quad (i=1, 2, \dots, N)$$

$$h_{i-1} = x_i - x_{i-1}, \quad h_i = x_{i+1} - x_i$$

$H_{i,j}$  is the hydraulic head at  $x=x_i, y=y_j$ . Since the computational domain is subdivided into rectangles, we have  $h_i=h$  ( $i=1, 2, \dots, N$ ),  $h$  is the spatial step in the  $x$  direction. Therefore,

$$\lambda_i = \frac{1}{2}, \quad \mu_i = \frac{1}{2}$$

$$g_{i,j} = -\frac{3}{2h} (H_{i+1,j} - H_{i-1,j}) \quad (i=1, 2, \dots, N)$$

At the same time, Eq. (4.1) can be further simplified as:

$$m_{i-1,j} + 4m_{i,j} + m_{i+1,j} = 2g_{i,j} \quad (i=1, 2, \dots, N) \quad (4.2)$$

Eq. (4.2) is a tridiagonal equation group,  $m_{i,j}$  ( $i=1, 2, \dots, N$ ) can be resolved by utilizing the Thomas algorithm.

As a result,  $\left(\frac{\partial H}{\partial x}\right)_{i,j} = m_{i,j}$  ( $i=1, 2, \dots, N-1$ )

at the boundary:

$$\left(\frac{\partial H}{\partial x}\right)_{0,j} = \frac{H_{1,j} - H_{0,j}}{h} = m_{0,j}$$

$$\left(\frac{\partial H}{\partial x}\right)_{N,j} = \frac{H_{N,j} - H_{N-1,j}}{h} = m_{N,j}$$

In similar manner, in the  $y$  direction, we can also obtain:

$$\left(\frac{\partial H}{\partial y}\right)_{i,j} = m_{i,j} \quad (j=1, 2, \dots, M-1)$$

at the boundary:

$$\left(\frac{\partial H}{\partial y}\right)_{i,0} = \frac{H_{i,1} - H_{i,0}}{h'}$$

$$\left(\frac{\partial H}{\partial y}\right)_{i,M} = \frac{H_{i,M} - H_{i,M-1}}{h'}$$

where  $h'$  is the spatial step in the  $y$  direction. So that  $v_x, v_y$  at the all grid nodes can be computed, and Eq. (3.1) may be consider as a two-dimensional hyperbolic equation with constant coefficients.

For equation (3.1), alternating direction implicit scheme is given as follows:  
in the  $x$  direction:

$$v_x^{(n)} \frac{H_{i+1,j}^{(n+1/4)} - H_{i-1,j}^{(n+1/4)}}{2\Delta x} + v_y^{(n)} \frac{H_{i,j+1}^{(n)} - H_{i,j-1}^{(n)}}{2\Delta y} = \frac{1}{2} \mu \frac{H_{i,j}^{(n+1/4)} - H_{i,j}^{(n)}}{\Delta t/4}$$

rewritling it as:

$$\frac{v_x^{(n)} \Delta t}{4h} H_{i-1,j}^{(n+1/4)} + \mu H_{i,j}^{(n+1/4)} - \frac{v_x^{(n)} \Delta t}{4h} H_{i+1,j}^{(n+1/4)}$$

$$= \mu H_{i,j}^{(n)} + \frac{v_y^{(n)} \Delta t}{4h'} (H_{i,j+1}^{(n)} - H_{i,j-1}^{(n)}) \quad (i=1, 2, \dots, N-1) \quad (4.3)$$

in the  $y$  direction:

$$\begin{aligned} & v_x^{(n+1/4)} \frac{H_{i+1,j}^{(n+1/4)} - H_{i-1,j}^{(n+1/4)}}{2\Delta x} + v_y^{(n+1/4)} \frac{H_{i,j+1}^{(n+1/2)} - H_{i,j-1}^{(n+1/2)}}{2\Delta y} \\ &= \frac{1}{2} \mu \frac{H_{i,j}^{(n+1/2)} - H_{i,j}^{(n+1/4)}}{\Delta t/4} \end{aligned}$$

just the same, we have

$$\begin{aligned} & \frac{v_y^{(n)} \Delta t}{4h'} H_{i,j-1}^{(n+1/2)} + \mu H_{i,j}^{(n+1/2)} - \frac{v_y^{(n)} \Delta t}{4h'} H_{i,j+1}^{(n+1/2)} \\ &= \mu H_{i,j}^{(n+1/4)} + \frac{v_x^{(n+1/4)} \Delta t}{4h} (H_{i+1,j}^{(n+1/4)} - H_{i-1,j}^{(n+1/4)}) \quad (j=1, 2, \dots, M-1) \quad (4.4) \end{aligned}$$

Eq. (4.3) and Eq. (4.4) are all tridiagonal equation groups.

At the initial time, the hydraulic heads  $H_{i,j}^{(n)} (i=1, 2, \dots, N)$  are described,  $H_{1,j}^{(n+1/4)}, H_{2,j}^{(n+1/4)}, \dots, H_{N-1,j}^{(n+1/4)}$  can be solved by the Thomas algorithm in the  $x$  direction, after that, substituting  $H_{i,j}^{(n+1/4)} (i=0, 1, \dots, N)$  into Eq. (4.4),  $H_{i,1}^{(n+1/2)}, H_{i,2}^{(n+1/2)}, \dots, H_{i,M-1}^{(n+1/2)}$  at  $(n+1/2)\Delta t$  moment are obtained by Thomas algorithm again, therefore, the hydraulic heads  $H_{i,j}^{(n+1/2)} (i=0, 1, \dots, N, j=0, 1, \dots, M)$  at  $(n+1/2)\Delta t$  moment in Eq. (3.1) can be obtained.

For equation (3.2), since it is a two-dimensional nonlinear partial differential equation, the alternating direction Picard iterative technique will be used.

The principle of Picard iteration is<sup>[2]</sup>: for the nonlinear algebraic equation group formed through discretization, constant and nonlinear parts were moved to the right side. An initial guess of the solution is required to evaluate the nonlinear terms, then a linear system of equations is solved. The result is substituted back into the right side, and the process is repeated. Iteration is not stopped until some convergence criterion has been met. Now, the solution is the result which is needed. Generally the derived solution at the preceding time step is acted as the initial value of evaluation.

$$\text{For Eq. (3.2):} \quad \kappa H \frac{\partial^2 H}{\partial x^2} + \kappa H \frac{\partial^2 H}{\partial y^2} = \frac{1}{2} \mu \frac{\partial H}{\partial t} \quad t \in \left( \left( n + \frac{1}{2} \right) \Delta t, (n+1) \Delta t \right)$$

its discreted form in the  $x$  direction can be written as

$$\begin{aligned} \frac{\mu}{2} \frac{H_{i,j}^{(n+3/4)} - H_{i,j}^{(n+1/2)}}{\Delta t/4} &= \kappa H_{i,j}^{(n+3/4)} \frac{H_{i-1,j}^{(n+3/4)} - 2H_{i,j}^{(n+3/4)} + H_{i+1,j}^{(n+3/4)}}{\Delta x^2} \\ &+ \kappa H_{i,j}^{(n+1/2)} \frac{H_{i,j-1}^{(n+1/2)} - 2H_{i,j}^{(n+1/2)} + H_{i,j+1}^{(n+1/2)}}{\Delta y^2} \end{aligned}$$

rearranging it as:

$$H_{i,j}^{(n+3/4)} = \frac{\kappa \Delta t}{2\mu h^2} H_{i,j}^{(n+3/4)} (H_{i-1,j}^{(n+3/4)} - 2H_{i,j}^{(n+3/4)} + H_{i+1,j}^{(n+3/4)})$$

$$+\frac{\kappa\Delta t}{2\mu h'^2}H_{i,j}^{(n+1/2)}(H_{i,j-1}^{(n+1/2)}-2H_{i,j}^{(n+1/2)}+H_{i,j+1}^{(n+1/2)})+H_{i,j}^{(n+1/2)} \quad (4.5)$$

In Eq. (4.5), the first term on the right side is nonlinear.  $H_{i-1,j}^{(n+1/2)}$ ,  $H_{i,j}^{(n+1/2)}$  and  $H_{i+1,j}^{(n+1/2)}$  which are considered as the evaluation values of  $H_{i-1,j}^{(n+3/4)}$ ,  $H_{i,j}^{(n+3/4)}$  and  $H_{i+1,j}^{(n+3/4)}$ , respectively, are substituted into the first term in Eq. (4.5), and the result of the estimation is computed. Since the second and third terms are known, the first time iterative value of  $H_{i,j}^{(n+3/4)}$ ,  $^{(1)}H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ) can be obtained by adding them together. The second time iterative value of  $H_{i,j}^{(n+3/4)}$ ,  $^{(2)}H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ), can be also solved, with the aid of substituting  $^{(1)}H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ) back into the first term on the right side of Eq. (4.5). This progress could not be stopped, until the difference between the  $m$ th and  $(m-1)$ th iterative value of  $H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ) has been less than the constant  $\varepsilon$  which are given, or  $|^{(m)}H_{i,j}^{(n+3/4)} - ^{(m-1)}H_{i,j}^{(n+3/4)}| < \varepsilon$  ( $\varepsilon$ , a positive constant, is little enough), then  $^{(m)}H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ) is the result of  $H_{i,j}^{(n+3/4)}$  ( $i=1, 2, \dots, N-1$ ). In the  $y$  direction:

$$\begin{aligned} \frac{\mu}{2} \frac{H_{i,j}^{(n+1)} - H_{i,j}^{(n+3/4)}}{\Delta t/4} &= \kappa H_{i,j}^{(n+1)} \frac{H_{i,j-1}^{(n+1)} - 2H_{i,j}^{(n+1)} + H_{i,j+1}^{(n+1)}}{\Delta y^2} \\ &\quad + \kappa H_{i,j}^{(n+3/4)} \frac{H_{i-1,j}^{(n+3/4)} - 2H_{i,j}^{(n+3/4)} + H_{i+1,j}^{(n+3/4)}}{\Delta x^2} \end{aligned}$$

rearranging it as

$$\begin{aligned} H_{i,j}^{(n+1)} &= \frac{\kappa\Delta t}{2\mu h'^2} H_{i,j}^{(n+1)} (H_{i,j-1}^{(n+1)} - 2H_{i,j}^{(n+1)} + H_{i,j+1}^{(n+1)}) \\ &\quad + \frac{\kappa\Delta t}{2\mu h'^2} H_{i,j}^{(n+3/4)} (H_{i-1,j}^{(n+3/4)} - 2H_{i,j}^{(n+3/4)} + H_{i+1,j}^{(n+3/4)}) \quad (4.6) \end{aligned}$$

The first term on the right side of Eq. (4.6) is also a nonlinear, when the iteration being analogous to the  $x$  direction is implemented,  $^{(1)}H_{i,j}^{(n+1)}$  ( $j=1, 2, \dots, M-1$ ), which will be solved is the solution to the whole problem at the  $(n+1)\Delta t$  moment, or  $H_{i,j}^{(n+1)}$  ( $j=1, 2, \dots, M-1$ ).

Summing up (4.3), (4.4), (4.5) and (4.6), we may obtained the solution,  $H_{i,j}^{(n+1)}$  ( $i=1, 2, \dots, N-1; j=1, 2, \dots, M-1$ ) at the  $(n+1)\Delta t$  moment. In regular order, phreatic heads at the prescribed moment  $T$  ( $T=NT \cdot \Delta t$ ,  $NT$  is the total number of time step) will be derived.

## V. Numerical Example

Some mine in Jiangxi Province is a recharge mine of fracture-karst water. In order to understand the mine inflow, the pumping test with multiple observation wells were finished from the end of 1991 to the earlier 1992. According to the conceptual model of research region, its mathematical model could be described as

$$\frac{\partial}{\partial x} \left( \kappa H \frac{\partial H}{\partial x} \right) + \frac{\partial}{\partial y} \left( \kappa H \frac{\partial H}{\partial y} \right) - \sum_{j=1}^m Q_j \delta(x-x_j, y-y_j) + W = \mu \frac{\partial H}{\partial t}$$

$$H(x, y, 0) = H_0(x, y)$$

$$H(x, y, t) \big|_{r_1} = H_1(x, y, t) \quad ((x, y) \in \Gamma_1)$$

$$\frac{\partial H(x, y, t)}{\partial n} \bigg|_{r_2} = 0 \quad ((x, y) \in \Gamma_2)$$

where  $W$ —other source/sink terms ( $L^3T^{-1}$ )

$Q_j$ —pumpage of the  $j$ th well ( $L^3T^{-1}$ )

$m$ —total number of pumping wells

$\delta(x-x_j, y-y_j)$ —two-dimensional Dirac function

the others are as before.

The area of computational domain is about  $60\text{km}^2$ , it is discretized into 465 rectangle elements and 653 nodes. At the same time, in terms of geologic and hydrogeologic conditions, 13 heterogeneous parameter subregions are divided. The whole computation may be made in two steps. At first, the model is rectified by using the pumping test data in the earlier stage, and then, by means of the pumping test data in the later stage, the numerical computation is carried out by employing the technique proposed in this paper and linearized finite difference method, respectively. At 310# node (G206# observation well), the computational results from the two methods mentioned above and the practical values are expressed in the comparative curve, Fig. 1. It may be seen, that the solutions obtained by the method suggested in this paper are more close to the practical values than the linearized finite difference method.

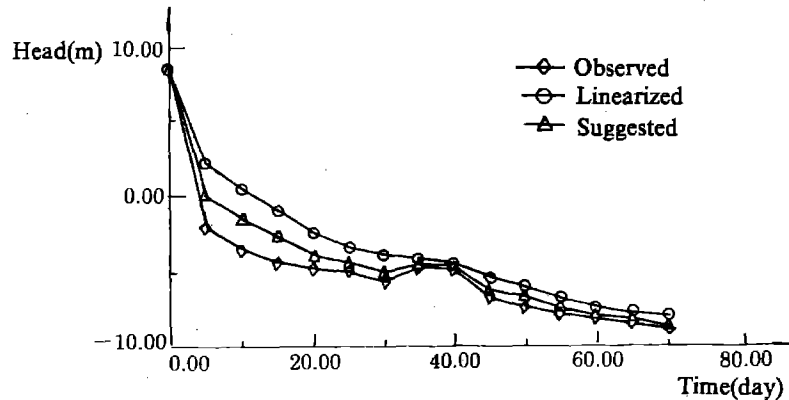


Fig. 1

## VI. Conclusion

The derivative process of the computational method suggested in this paper which is used to solve two-dimensional phreatic flow equation is slightly prolix, but, in the process of computation, by means of operator splitting, the alternating direction method is used, so, the numerical stability is guaranteed. Especially, since only some of simple tridiagonal equation groups are solved, the computational code can be made easy, the computer resource may be also saved and the computational efficiency is enhanced. Furthermore, from the results of numerical example, the accuracy is improved in comparison with the conventional linearized technique. Therefore, it will become worthy of the reference for solving the similar nonlinear problems.

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