Applied Mathematics and Mechanics
(English Edition, Vol. 16, No. 10, Oct. 1995)

Published by SU,
Shanghai, China

INVARIANT MANIFOLDS AND THEIR STABILITY IN A THREE-DIMENSIONAL MEASURE-PRESERVING MAPPING SYSTEMS

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(Received Dec. 1, 1994; Communicated by Chou Huanwen)

Abstract

In the paper researches on a three-dimensional measure-preserving mapping system are made, which is the three-dimensional extension of the Keplerian mapping. With the formal series method the expressions of the invariant curves and invariant tori are obtained. Finally the stability of these invariant manifolds is also discussed.

Key words measure-preserving mapping, chaos, invariant manifold, stability

I. Introduction

Three-dimensional measure-preserving mapping systems have special and rich dynamical behaviors which are much different from those of even dimensional systems[1-3]. The linearized mapping near the fixed point has one real eigenvalue at least. Additionally, the product of all eigenvalues is unit. So there usually exist some eigenvalues whose modules are larger than one. Therefore the fixed points usually are not linearly stable, which is much different from that of two-dimensional case. For three-dimensional measure-preserving mapping, there usually do not exist one-dimensional or two-dimensional invariant manifolds in the neighbourhood of a fixed point, but two-dimensional invariant manifolds do exist in the neighbourhood of the one-dimensional invariant manifolds[2]. Then for the three-dimensional mapping system study on the invariant manifolds and their stability is of importance. The paper [3] gives a sufficient criterion of the unstabilizing of the one-dimensional invariant manifolds. In paper [4] we find there must exist other structures for unstabilizing of these invariant manifolds. Deeply discussing on the problem depends on the expressions of the invariant manifolds. In this paper we will give the precise expressions of these invariant manifolds in formal series and discuss their stability in detail.

As well-known, a $2n$-dimensional Hamiltionian system may be simplified to an even-dimensional measure-preserving mapping through Poincare section and energy section. Then is the study on odd-dimensional mapping systems unpractical? In fact the three-dimensional mapping systems have been used to study many complicated system, such as Couette-Taylor fluids and the comet's motion[4-5]. In paper [4] we derived a three-dimensional mapping which reflects the motion of the near-parabolic orbital comet. In this paper we research into the invariant manifolds and their stability of that mathematical model. The methods may be extended to other three-dimensional measure-presserving mapping system.

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II. Mapping System

In paper [4] we have studied a problem on celestial body's motion—the comet with near parabolic orbit. A mapping has been derived,

$$\left.\begin{aligned} &K'=K+\mu\Big(\sum_{i=1}^{7}b_i\sin ig+\rho\sin\omega\cos g\Big) \\ &g'=g+2\pi/(-K')^{3/2} \qquad \mathrm{mod}(2\pi) \\ &\omega'=\omega+\mu\sum_{i=0}^{7}a_i\cos ig \qquad \mathrm{mod}(2\pi) \end{aligned}\right\} \tag{2.1}$$

where K, g and ω denote the comet's energy, phase angle and perihelion longitude respectively; ρ is the parameter of primary body's eccentricity; a_i and b_i are constant parameter; μ is small parameter. With neglecting the high frequency terms in perturbation effects for convenience the simplified mapping is deduced as follows,

$$M_3:\ \begin{cases} K'=K+\mu R(g,\ \omega) \\ g'=g+2\pi/(-K')^{3/2} \qquad \mathrm{mod}(2\pi) \\ \omega'=\omega+\mu(\alpha+\cos g) \qquad \mathrm{mod}(2\pi) \end{cases} \tag{2.2}$$

where α is parameter and

$$R(g,\ \omega)=\sin g+\rho\cos g\sin\omega \tag{2.3}$$

When ρ is zero, system M_3 degenerates to a two-dimensional mapping which is known as the Keplerian mapping[6]

$$M_2:\ \begin{cases} K'=K+\mu\sin g \\ g'=g+2\pi/(-K')^{3/2} \qquad \mathrm{mod}(2\pi) \end{cases} \tag{2.4}$$

For

$$\det\left|\frac{\partial(K',g',\omega')}{\partial(K,g,\omega)}\right|=\det\left|\frac{\partial(K',g')}{\partial(K,g)}\right|=1$$

systems M_3 and M_2 are both measure preserving. Additionally M_3 is the measure-preserving extension of M_2.

There exist 1-periodic solutions for system M_2,

$$\left.\begin{aligned} &g^*=0,\ \pi \\ &K^*=-(1/m)^{2/3} \end{aligned}\right\} \tag{2.5}$$

where m is a positive integer.

The primary body's period is 2π in scaling time unit. The comet's period will be $2m\pi$ while its energy equals to K^*. Then m:1 resonance occurs. Let $\Theta=D(K',g')/D(K,g)$, the trace of the matrix will be

$$\mathrm{Trace}\Theta=2+3\pi\mu\cos g/(-K)^{5/2} \tag{2.6}$$

When $|\mathrm{Trace}\Theta|_{(K^*,g^*)}$ is greater than two, the corresponding 1-periodic solution will be unstable[7]. Therefore we have conclusions as follows. The 1-periodic solutions with $g^*=0$ are unstable hyperbolic fixed points. For the 1-periodic solution with $g^*=\pi$, only when $K^*<-(3\pi\mu/4)^{2/5}$ it is a stable elliptic fixed point.

III. The Calculation of One-Dimensional Invariant Curves

Let $(K(\omega), g(\omega))$ be the one-dimensional invariant curves of the system M_3, according to its definition it satisfies the following equations,

$$\left.\begin{aligned} &K(\omega')=K(\omega)+\mu R(g(\omega),\ \omega) \\ &g(\omega')=g(\omega)+2\pi/(-K(\omega'))^{3/2} \qquad \mathrm{mod}(2\pi) \\ &\omega'=\omega+\mu[a+\cos g(\omega)] \qquad \mathrm{mod}(2\pi) \end{aligned}\right\} \tag{3.1}$$

The solutions of above functional equations are expanded in formal series as follows,

$$\left.\begin{aligned} &K(\omega)=K_0(\omega)+\mu K_1(\omega)+\mu^2K_2(\omega)+\cdots \\ &g(\omega)=g_0(\omega)+\mu g_1(\omega)+\mu^2 g_2(\omega)+\cdots \end{aligned}\right\} \tag{3.2}$$

Then following series expansions are got

$$\begin{aligned} K(\omega')=K(\omega)+\mu\frac{dK_0}{d\omega}(a+\cos g_0)+\mu^2\Big[\frac{dK_0}{d\omega}(-g_1\sin g_0) \\ +\frac{dK_1}{d\omega}(a+\cos g_0)+\frac{1}{2}\frac{d^2K_0}{d\omega^2}(a+\cos g_0)^2\Big]+\cdots \end{aligned} \tag{3.3}$$

$$\begin{aligned} g(\omega')=g(\omega)+\mu\frac{dg_0}{d\omega}(a+\cos g_0)+\mu^2\Big[-\frac{dg_0}{d\omega}(g_1\sin g_0) \\ +\frac{dg_1}{d\omega}(a+\cos g_0)+\frac{1}{2}\frac{d^2g_0}{d\omega^2}(a+\cos g_0)^2\Big]+\cdots \end{aligned} \tag{3.4}$$

otherwise, $2\pi(-K')^{3/2}$ is expanded in following form

$$\begin{aligned} 2\pi/(-K')^{3/2}=\frac{2\pi}{(-K_0)^{3/2}}\Big\{1-\frac{3}{2}\Big(\frac{K_1+r_1}{K_0}\Big)\mu+\Big[\frac{15}{8}\Big(\frac{K_1+r_1}{K_0}\Big)^2 \\ -\frac{3}{2}\Big(\frac{K_2+r_2}{K_0}\Big)\Big]\mu^2+\cdots\Big\} \end{aligned} \tag{3.5}$$

where

$$r_1=\sin g_0+\rho\sin\omega\cos g_0$$
$$r_2=\cos g_0-\rho\sin\omega\sin g_0$$

With (3.3), (3.4) and (3.5) employed in the functional equations and the coefficients of terms with the same order in μ compared, the solutions of any order are obtained

in zeroth order

$$K_0(\omega)=-(1/m)^{2/3} \tag{3.6}$$

in the first order

$$\frac{dK_0}{d\omega}(a+\cos g_0)=r_1,\qquad \frac{dg_0}{d\omega}(a+\cos g_0)=\frac{3\pi(K_1+r_1)}{(-K_0)^{5/2}} \tag{3.7}$$

the solutions of above equaitons are

$$\left.\begin{aligned} g_0(\omega) &= \pi - \mathrm{arctg}(\rho\sin\omega) \\ K_1(\omega) &= \frac{(-K_0)^{5/2}}{3\pi}[a+\cos g_0(\omega)]\frac{dg_0(\omega)}{d\omega} \end{aligned}\right\} \tag{3.8}$$

in the second order

$$\left.\begin{aligned} g_1(\omega) &= \frac{dK_1}{d\omega}(a+\cos g_0)/(\cos g_0-\rho\sin\omega\sin g_0) \\ K_2(\omega) &= \frac{(-K_0)^{5/2}}{3\pi}\left\{-\frac{dg_0}{d\omega}g_1\sin g_0+\frac{dg_1}{d\omega}(a+\cos g_0)\right. \\ &\quad \left.+\frac{1}{2}\frac{d^2g_0}{d\omega^2}(a+\cos g_0)^2-\frac{15\pi}{4(-K_0)^{7/2}}(K_1+r_1)\right\}-r_2 \end{aligned}\right\} \tag{3.9}$$

Through continuing the process the solutions to higher order, such as (K_3, g_2), (K_4, g_3), ..., (K_{n}, g_{n-1}) will be obtained.

Those expressions of high order derivatives used above are listed as follows:

$$dg_0/d\omega = -\rho\cos\omega/(1+\rho^2\sin^2\omega) \tag{3.10}$$

$$d^2g_0/d\omega^2 = [\rho\sin\omega+\rho^3\sin\omega(1+\cos^2\omega)]/(1+\rho^2\sin^2\omega)^2 \tag{3.11}$$

$$\begin{aligned} d^3g_0/d\omega^3 &= [\rho\cos\omega+\rho^3(\cos\omega+\cos^3\omega-2\cos\omega\sin^2\omega)](1+\rho^2\sin^2\omega)^2 \\ &\quad -4\rho^2\sin\omega\cos\omega[\rho\sin\omega+\rho^3\sin\omega(1+\cos^2\omega)]/(1+\rho^2\sin^2\omega)^3 \end{aligned} \tag{3.12}$$

$$\frac{dK_1}{d\omega} = \frac{(-K_0)^{5/2}}{3\pi}\left[(-\sin g_0)\left(\frac{dg_0}{d\omega}\right)^2+(a+\cos g_0)\frac{d^2g_0}{d\omega^2}\right] \tag{3.13}$$

$$\begin{aligned} \frac{d^2K_1}{d\omega^2} &= \frac{(-K_0)^{5/2}}{3\pi}\left\{(-\cos g_0)\left(\frac{dg_0}{d\omega}\right)^3-3\sin g_0\frac{dg_0}{d\omega}\frac{d^2g_0}{d\omega^2}\right. \\ &\quad \left.+(a+\cos g_0)\frac{d^3g_0}{d\omega^3}\right\} \end{aligned} \tag{3.14}$$

$$\begin{aligned} \frac{dg_1}{d\omega} &= -\frac{d^2K_1}{d\omega^2}(a+\cos g_0)(1+\rho^2\sin^2\omega)^{-1/2} \\ &\quad +\frac{dK_1}{d\omega}\frac{dg_0}{d\omega}\sin g_0(1+\rho^2\sin^2\omega)^{-1/2} \\ &\quad +\frac{dK_1}{d\omega}(a+\cos g_0)\rho^2\sin\omega\cos\omega(1+\rho^2\sin^2\omega)^{-3/2} \end{aligned} \tag{3.15}$$

With above method the expressions of the one-dimensional invariant manifolds in the form of the power series can be obtained.

When ρ equals zero, the points $(g=\pi,\ K=-(1/m)^{2/3})$ are the centres of the resonant zones. As ρ increases these centres shifts both along g axis and along K axis. Generally, the width of the shift along g is of order $o(1)$ and gets its maximum values at $\omega=\pi/2,\ 3\pi/2$. The width of the shift along K is of order $o(\mu)$ and gets its maximum values at $\omega=0,\ \pi$. It is interesting that the parameter α affects the shifts of the centres. We find that these shifts widths increase in both directions with increasing α. The shift width of 1/1 resonant zone is larger than that of other resonant zone.

IV. The Calculation of Two-Dimensional Invariant Tori

For the three-dimensional measure preserving mapping system there must exist the two-dimensional invariant manifolds in the neighborhood of the one-dimensional invariant manifolds[2]. Due to the fact that a continuous Hamiltonian system may be discretized into a measure-preserving mapping through the Poincare section method, the above invariant manifolds correspond the local invariant integrals of the continuous system.

Let $K=K(g, \omega, \mu)$ be the invariant torus of the system M_α, we try to expand it in the formal series of small parameter μ and get null results. In paper [8] we have estimated the half width of $m/1$ resonant zones in mapping system and find it being of order $\mu^{1/2}$. The fact implies that the function $K(g, \omega, \mu)$ should be expanded in the formal series of $\mu^{1/2}$.

$$K(g, \omega, \mu)=K_0(g, \omega)+\mu^{1/2}K_1(g, \omega)+\mu K_2(g, \omega)+\cdots \tag{4.1}$$

According to the definition of the invariant torus the smooth function $K(g, \omega, \mu)$ satisfies following functional equation

$$K(g', \omega', \mu)=K(g, \omega, \mu)+\mu R(g, \omega) \tag{4.2}$$

Substituting the formal series solution (4.1) and the expression (2.2) into the Eq. (4.2) and comparing the coefficients of the same order terms in both sides of the equations, one gets that

in zeroth order

$$K_0(g, \omega)=-(1/m)^{2/3} \tag{4.3}$$

in the $\frac{1}{2}$th order

$$\frac{3\pi}{(-K_0)^{5/2}}K_1\frac{\partial K_0}{\partial g}+K_1\left(g+\frac{2\pi}{(-K_0)^{3/2}}, \omega\right)=K_1(g, \omega) \tag{4.4}$$

Because (4.3) is tenable any new information is not got from above equation.

in the first order

$$\left\{\frac{1}{2}\left[\frac{3\pi}{(-K_0)^{5/2}}K_1\right]^2\frac{\partial^2 K_0}{\partial g^2}+\frac{\partial K_0}{\partial \omega}(a+\cos g)\right\}+\frac{3\pi K_1}{(-K_0)^{5/2}}\frac{\partial K_1}{\partial g}$$
$$+K_2\left(g+\frac{2\pi}{(-K_0)^{3/2}}, \omega\right)=K_2(g, \omega)+R(g, \omega) \tag{4.5}$$

then

$$\frac{3\pi}{(-K_0)^{5/2}}K_1\frac{\partial K_1}{\partial g}=R(g, \omega) \tag{4.6}$$

the above differential equation has the solution in the following form

$$\frac{1}{2}K_1^2(g, \omega)=\frac{(-K_0)^{5/2}}{3\pi}(-\cos g+\rho\sin g\sin\omega)+U(\omega) \tag{4.7}$$

where $U(\omega)$ is a function awaiting determination.

Let us make variables transformations as follows.

$$\pi\left\{\begin{array}{l}\bar{K}=K-K(\omega)\\ \bar{g}=g-g(\omega)\\ \bar{\omega}=\omega\end{array}\right. \tag{4.8}$$

where $(K(\omega), g(\omega))$ is the expression of the invariant curve.

It is easy to prove that the mapping $\bar{M}_3=\pi M_3\pi^{-1}$ is measure preserving. Therefore to $o(\mu)$ order the $\bar{M}_3$ is also measure preserving in the new variables $(\bar{K}_1, \bar{g}, \omega)$.

Being expanded in the vicinity of the one-dimensional invariant manifold, Eq. (4.7) approximates to an elliptic equation as follows

$$\frac{1}{2}\bar{K}_1^2+\frac{1}{2}q(\omega)\bar{g}^2=q(\omega)+U(\omega) \tag{4.9}$$

where

$$q(\omega)=(-K_0)^{5/2}\sqrt{1+\rho^2\sin^2\omega}/3\pi$$

The elliptic area is

$$A(\omega)=2\pi[1+U(\omega)/q(\omega)]\sqrt{q(\omega)} \tag{4.10}$$

Approximately the variations of ω' and ω satisfy the relation

$$\delta\omega'=\left[1+\frac{(-K_0)^{5/2}}{3\pi}\frac{q'(\omega)}{q^2(\omega)}\mu\right]\delta\omega, \qquad q'(\omega)=\frac{dq(\omega)}{d\omega} \tag{4.11}$$

For system $\bar{M}_3$ is measure-preserving, the equation $A(\omega')\delta\omega'=A(\omega)\delta\omega$ be satisfied. The function $U(\omega)$ is derived from above relation.

$$U(\omega)=cq^{3/2}(\omega)/[aq(\omega)-(-K_0)^{5/2}/3\pi]-q(\omega) \tag{4.12}$$

where c is an integral constant.

With the analytic expressions of the invariant tori we know well their geometric characters. When the parameter α is greater than one, the section area $A(\omega)$ gets its maximum values and minimum values at $\omega=0$, π and $\omega=\pi/2$, $3\pi/2$ respectively. When $0<\alpha<1$, the area gets its maximum values and minimum values at $\omega=\pi/2$, $3\pi/2$ and $\omega=0$, π respectively. The ratio of the maximum area to the minimum area is

$$A_{max}/A_{min}=(\alpha-1/\sqrt{1+\rho^2})/(\alpha-1) \qquad \alpha>1 \tag{4.13}$$

$$A_{max}/A_{min}=(1-\alpha)/(1/\sqrt{1+\rho^2}-\alpha) \qquad 0<\alpha<1 \tag{4.14}$$

We find that the invariant tori have similar structure with respect to different resonant zones. Its shape looks like a water pipe. The degree of the pipe's thickness varies with ω. The Eqs. (4.13) and (4.14) describe the characters of the pipe's shape.

V. The Stability of the m/1 Resonant Zones

When the parameter ρ is small, the $m/1$ ($m=1, 2, ..., 6$) resonant zones are stable. There are one-dimensional and two-dimensional invariant manifolds formularized by Eq. (3.2) and Eq. (4.1) for the system M_3. With increasing the parameter ρ the invariant tori will be destroyed from exterior to interior successively. When ρ is larger than a critical value ρ_c, the one-dimensional invariant manifold will be destroyed and the counterpart m/l resonant zone will be unstable.

The changes on ω along the invariant curves are derived from the expression (3.2).

$$\Delta\omega=\omega'-\omega=\mu(a-1/\sqrt{1+\rho^2\sin^2\omega})+o(\mu^2) \tag{5.1}$$

When $\rho\geqslant\rho_c=\sqrt{1/a^2-1}$, there exist solutions for $\Delta\omega=0$

$$\omega^*=\beta,\ \pi+\beta,\ \pi-\beta,\ 2\pi-\beta \tag{5.2}$$

where $\beta=\arcsin\sqrt{1/a^2-1}/\rho$. In this case the ($g^*=g(\omega^*)$, $K^*=K(\omega^*)$, ω^*) are the fixed points of the mapping M_λ. It is easy to prove that these fixed points be unstable. That is, assuming $\rho\geqslant\rho_c=\sqrt{1/a^2-1}$ the invariant curves degenerate into the unstable fixed points. In the terms of expression (4.14), we find that, when the parameter is equal to the critical value, the A_{mn} tends to infinite at $\omega^*=3\pi/2$. This means that the diffusion phenomenon occurs on the section plane.

In a addition to above discussions, there exists other different unstable mechanism for the case $\alpha>1$.

The characteristic matrix of the mapping $\Theta=\partial(K',\ g')/\partial(K,\ g)$. The absolute value of the trace of the matrix Θ are derived as follows

$$L=|\mathrm{Trace}\Theta|=\left|2+3\pi\mu\,\frac{\partial R(g,\omega)}{\partial g}\Big/(-K-\mu R(g,\ \omega))^{5/2}\right| \tag{5.3}$$

Substituting the expressions of the invariant curves (3.2) into above equation one gets that

$$L(\omega)=|2-3\pi\mu m^{5/3}\sqrt{1+\rho^2\sin^2\omega}| \tag{5.4}$$

The critical parameter ρ is estimated according to the fact that the supremum of the function $L(\omega)$ being greater than two results in unstability.

$$\rho_c=\left[\left(\frac{4}{3\pi\mu m^{5/3}}\right)^2-1\right]^{\frac{1}{2}} \tag{5.5}$$

It is well-known that the Liapunov Characteristic Numbers (LCNs) is a good indicator to the regular or stochastic motions. With the LCNs method we calculate the critical value numerically and compare these computational results with our theoretical estimations (Table 1).

Table 1 ($\mu=0.01,\ a=1.1$)

Resonant zones	2/1	3/1	4/1	5/1	6/1
Computational results	13.4	6.5	4.3	2.7	1.8
Theoretical results	13.3	6.7	4.1	2.7	1.9

The maximum shift of the centres for the resonant zones are formularized as follows

$$\delta K=\mu(a-1)\rho/3\pi m^{5/3}+o(\mu^2) \tag{5.6}$$

$$\delta g=\mathrm{arctg}\rho+\mu\rho(a-1/\sqrt{1+\rho^2})^2/(3\pi m^{5/3})(1+\rho^2)^{3/4}+o(\mu^2) \tag{5.7}$$

Interestingly the parameter α also affects the shift of the centres of the resonant zones and influences our estimations. To discuss the problem in detail we take the 2/1 resonance case for example. With LCNs methods we calculate the relations between the critical values and the parameter α (Table 2). When the value of the parameter α comes close to one, δK tends to zero and the critical value is 13.4 which is agreeable with our theoretical reuslts. But when parameter α increases, the shift along the K direction δK increases too large and the critical value deviates from our theoretical estimations.

Table 2

($\mu=0.01$)

Par. α	1.1	2.0	3.0	3.5	4.0
Cri. ρ_c	13.4	13.4	11.1	8.5	6.7
δK	4.46×10^{-4}	4.79×10^{-3}	9.37×10^{-3}	8.93×10^{-3}	8.34×10^{-3}
δg	1.50	1.50	1.48	1.45	1.42

VI. Conclusion Remarks

In the paper the expressions of the one-dimesional and two-dimensional invariant manifolds corresponding to m 1 resonant zones are obtained through the formal series method. According to the analytic expressions the geometric characters of the invariant curves and tori are discussed in detail. We also study the stability of the m 1 resonant zones and obtain the sufficient criterions of the unstabilization of the resonant zones. The fact that there exist several structures for the unstabilization of the invariant curves is confirmed. As parameter α is smaller than one, or is greater than and comes close to one, the computations are agreeable to our theoretical results. With α increasing we conjecture that the overlap of the nearby resonant zones occurs. This effect may decrease the stability of the resonant zones and results in the deviation in our theoretical estimations. This is an open and interesting problem. It will be discussed in detail in the future paper.

Acknowledgment The author thanks Prof. Sun Yisui and Prof. Chen Shigang for helpful discussions.

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