

MULTIPLICITY RESULTS FOR A FOURTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract

This paper deals with multiplicity results for nonlinear elastic equations of the type

$$\begin{aligned} y^{(4)} - \alpha_1 y + \beta_1 y^3 + g(x, y, y') &= e, \quad 0 < x < 1 \\ y(0) = y''(0) = y'(1) = y'''(1) &= 0 \end{aligned}$$

where $e \in L^1(0, 1)$, $g: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a bounded continuous function, and the pair (α, β) satisfies

$$\alpha_1 + (0 + 0.5)^2 \pi^2 \beta_1 = (0 + 0.5)^4 \pi^4$$

and

$$\alpha_1 + (k + 0.5)^2 \pi^2 \beta_1 \neq (k + 0.5)^4 \pi^4, \quad \text{for all } k \in \mathbb{N}$$

Key words elastic beam, two-parameter eigenvalue problem, multiplicity result

1. Introduction

The static deformations of an elastic beam with one of its end simply supported and the other end clamped by sliding clamps are described by the following fourth-order boundary value problem

$$y^{(4)} + f(x)y = e(x), \quad 0 < x < 1 \quad (1.1)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.2)$$

In [3, 4], Gupta studied the following nonlinear analogue of the boundary value problem (1.1)–(1.2)

$$y^{(4)} - f(x, y, y', y'', y''') = e(x), \quad 0 < x < 1 \quad (1.3)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.4)$$

He proved several existing theorems under conditions on f that are related to the linear eigenvalue problem

$$y^{(4)} - \alpha y = 0 \quad (1.5)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.6)$$

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and

$$y^{(4)} + \beta y'' = 0 \quad (1.7)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.8)$$

In this paper we study the following nonlinear problem

$$y^{(4)} - \alpha_1 y + \beta_1 y'' + g(x, y, y'') = e, \quad 0 < x < 1 \quad (1.9)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.10)$$

where $e \in L^2(0, 1)$, $g: [0, 1] \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is a bounded continuous function, and the pair (α_1, β_1) satisfies

$$\alpha_1 + (0 + 0.5)^2 \pi^2 \beta_1 = (0 + 0.5)^4 \pi^4 \quad (1.11)$$

and

$$\alpha_1 + (k + 0.5)^2 \pi^2 \beta_1 \neq (k + 0.5)^4 \pi^4, \quad \text{for all } k \in \mathbf{N} \quad (1.12)$$

We obtain results on nonexistence, existence and multiplicity of solution of (1.9)–(1.10). Our method combines the well-known Lyapunov-Schmidt procedure with a connectivity properties of the solution set of parametrized families of compact vector fields.

In section II of this paper we study the two-parameter eigenvalue problem

$$y^{(4)} - \alpha y + \beta y'' = 0, \quad 0 < x < 1 \quad (1.13)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (1.14)$$

which generalizes (1.5)–(1.6) and (1.7)–(1.8). In section III we state the main result, while in section IV we provide the proof.

II. Two-Parameter Eigenvalue Problem

We begin this section by solving the eigenvalue problem (1.13)–(1.14). A pair (α, β) such that (1.13)–(1.14) possesses a nontrivial solution will be called an eigenvalue pair. A corresponding nontrivial solution will be eigenfunction.

Proposition 2.1 (α, β) is an eigenvalue pair of (1.13)–(1.14) if and only if

$$\alpha + (k + 0.5)^2 \pi^2 \beta = (k + 0.5)^4 \pi^4 \quad (2.1)$$

for some $k \in \mathbf{N} \cup \{0\}$.

Proof Define a linear operator $F: D(F) \rightarrow L^2(0, 1)$ by setting

$$D(F) = \{u \in L^2(0, 1) \mid u, u' \in AC[0, 1], u'' \in L^2(0, 1), u(0) = u'(1) = 0\} \quad (2.2)$$

and for $u \in D(F)$

$$F(u) = u'' \quad (2.3)$$

(Here $AC[0, 1]$ denotes the space of real value absolutely continuous functions on $[0, 1]$). Then

$$y^{(4)} + \beta y'' - \alpha y = (F + r_1)(F + r_2)y$$

For some $r_1, r_2 \in \mathbf{C}$. It is easy to see that if (1.13)–(1.14) possesses a nontrivial solution, then either $r_1 = (k + 0.5)\pi$ or $r_2 = (k + 0.5)\pi$ for some $k \in \mathbf{N} \cup \{0\}$. In either case, $\sin(k + 0.5)\pi x$ is a nontrivial solution of (1.13)–(1.14). By substituting this solution into (1.13), (2.1) follows. Conversely, if (2.1) holds, then clearly $\sin(k + 0.5)\pi x$ is a nontrivial solution of (1.13)–(1.14).

Next, for $j \in \mathbb{N} \cup \{0\}$, let us set

$$L_j = \{(\alpha, \beta) \mid \alpha + (j+0.5)^2 \pi^2 \beta = (j+0.5)^4 \pi^4\} \quad (2.4)$$

In view of the above proposition, we call L_j an eigenline of (1.13)–(1.14). We note that an eigenvalue pair (α, β) can belong to at most two eigenlines. If (α, β) belong to just one L_j , then the corresponding eigenspace is that spanned by $\sin(j+0.5)\pi x$. If (α, β) belong to $L_j \cap L_k$, then the corresponding eigenspace is that spanned by $\sin(k+0.5)\pi x$ and $\sin(j+0.5)\pi x$.

Suppose now that the pair $(\alpha_1, \beta_1) \in L_0 \setminus \bigcup_{n=1}^{\infty} L_n$. Let us define a linear operator $L: D(L) \subset L^2(0,1) \rightarrow L^2(0,1)$ by setting

$$D(L) = \left\{ u \in L^2(0,1) \mid \begin{array}{l} u', u'', u''' \in AC[0,1], u^{(iv)} \in L^2(0,1) \\ u(0) = u''(0) = u'(1) = u'''(1) = 0 \end{array} \right\} \quad (2.5)$$

and for $y \in D(L)$,

$$Ly = y^{(iv)} - \alpha_1 y + \beta_1 y'' \quad (2.6)$$

Then

$$H = L^2(0,1)$$

admits the (orthogonal) direct sum decomposition $H = V \oplus V^\perp$, where $V = \ker(L) = \text{span}\{\sin(\pi x/2)\}$ and $V^\perp = \text{Range}(L)$. Let

$$\phi(x) = \sqrt{2} \sin \frac{\pi x}{2}, \quad 0 < x < 1 \quad (2.7)$$

then $\|\phi\|_H = 1$ and we can write $y = s\phi + w$ and $e = t\phi + h$ for given $y \in D(L)$ and $e \in H$. Denote the orthogonal projections over V and V^\perp by P and Q , respectively.

Now, from, for example, the Fredholm Alternative it follows that the boundary value problem

$$y^{(iv)} - \alpha_1 y + \beta_1 y'' = b(x), \quad 0 < x < 1 \quad (2.8)$$

$$y(0) = y''(0) = y'(1) = y'''(1) = 0 \quad (2.9)$$

has a unique solution $y \in V^\perp$, for each $b \in V^\perp$. Moreover, this solution admits a Fourier series expansion of form

$$y(x) = \sum_{k=1}^{\infty} \frac{b_k \sin(k+0.5)\pi x}{(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1} \quad (2.10)$$

where

$$b(x) = \sum_{k=1}^{\infty} b_k \sin(k+0.5)\pi x \quad (2.11)$$

Also, we have that

$$y''(x) = - \sum_{k=1}^{\infty} \frac{b_k (k+0.5)^2 \pi^2 \sin(k+0.5)\pi x}{(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1} \quad (2.12)$$

From (2.11) and (2.12) we can easily see that the operators $A, B: V^\perp \rightarrow V^\perp$ define by

$$A(b) = g, \quad B(b) = g' \quad (2.13)$$

compact linear operators. In (2.13), $g \in V^\perp$ is the solution of (2.8)–(2.9) corresponding to $b \in V^\perp$. The norm of A and B are respectively given by

$$\|A\| = \max_{k \in \mathbb{N}} \frac{1}{|(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1|} \quad (2.14)$$

$$\|B\| = \max_{k \in \mathbb{N}} \frac{(k+0.5)^2 \pi^2}{|(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1|} \quad (2.15)$$

III. The Main Result

Let $e \in L^2(0,1)$ and $g: [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

Theorem 3.1 Suppose

(H1) g be a bounded function, i. e., there exists a constant $M > 0$ such that

$$|g(x, y, z)| \leq M, \quad \forall x \in [0,1], y, z \in \mathbb{R} \quad (3.1)$$

(H2) there is a constant $y_1 > 0$ such that

$$g(x, y_1, z) < 0 \quad \forall x \in [0,1], z < 0 \quad (3.2)$$

(H3)

$$\begin{aligned} g(x, y, z) &\leq 0 & \forall x \in [0,1], y > 0, z < 0, \\ g(x, y, z) &\geq 0 & \forall x \in [0,1], y < 0, z > 0. \end{aligned}$$

(H4)

$$\lim_{y^2+z^2 \rightarrow \infty} g(x, y, z) = 0 \quad \text{uniformly for } x \in [0,1]$$

(H5) there are positive constant a and b such that

$$a \max_{k \in \mathbb{N}} \frac{1}{|(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1|} + b \max_{k \in \mathbb{N}} \frac{(k+0.5)^2 \pi^2}{|(k+0.5)^4 \pi^4 - \alpha_1 - (k+0.5)^2 \pi^2 \beta_1|} < 1 \quad (3.3)$$

and

$$|g(x, y, z) - g(x, y_1, z)| \leq a|y - y_1| + b|z - z| \quad (3.4)$$

for all $x \in [0,1]$ and $y, y_1, z, z_1 \in \mathbb{R}$. Then there exist $-\infty < \tau_1 < 0$ and $0 \leq \tau_2 < +\infty$ such that (1.9)–(1.10) has at least one solution if and only if $t \in [\tau_1, \tau_2]$.

Moreover if $t \in (\tau_1, 0) \cup (0, \tau_2)$, then (1.9)–(1.10) has at least two distinct solutions.

Example Let $(\alpha_1, \beta_1) = (\pi^4/2^4, 0)$, we have that

$$\|A\| = \frac{1}{5\pi^4}, \quad \|B\| = \frac{9}{20\pi^2}$$

Let g_0 be defined by

$$g_0(x, y, z) = \frac{z}{1+y^2+z^2} \quad (3.5)$$

We have, for every $y, z, \eta, \xi \in \mathbf{R}$,

$$|g_0(x, \eta, z) - g_0(x, y, z)| \leq |\eta - y| + 3|z - z|$$

Hence, g_0 satisfies (H5). Clearly, the function g_0 satisfies (H1)–(H4). Thus we give an example to which Theorem 3.1 can be applied.

IV. Proof of the Theorem

Before proving Theorem 3.1 we present some preliminary results. The following lemma by Costa and Goncalves [1, Theorem 0] is the most important in the proof of our result.

Lemma 4.1 Let C be a bounded closed convex set in Banach space X and $K: [\alpha, \beta] \times C \rightarrow C$, $\alpha < \beta$, a compact mapping. Then the set

$$S_{\alpha, \beta} = \{(s, x) \in [\alpha, \beta] \times C \mid K(s, x) = x\} \quad (4.1)$$

contains a component $C_{\alpha, \beta}$ which connects $\{\alpha\} \times C$ and $\{\beta\} \times C$.

Let us define a (nonlinear) mapping $G: D(L) \subset H \rightarrow H$ by setting

$$(Gy)(x) = g(x, y(x), y''(x)), \quad x \in [0, 1] \quad (4.2)$$

then G uniformly bounded and continuous^{3, 4}. With this consideration, (1.9)–(1.10) can be written in the form of the following equation in H

$$Ly + G(y) = t\phi + h, \quad y \in D(L) \quad (4.3)$$

Now to solve the Eq. (4.3) it suffices to solve the system of equations

$$w + AQG(s\phi + w) = Ah \quad (4.4)$$

$$t'G(s\phi + w) = t\phi \quad (s \in \mathbf{R}, w \in D(L) \cap V^\perp) \quad (4.5)$$

Denote by $S \subset \mathbf{R} \times V^\perp$ the set of solution of (4.4), i. e.

$$S = \{(s, w) \in \mathbf{R} \times V^\perp \mid w \in D(L), w = A[h - QG(s\phi + w)]\} \quad (4.6)$$

Clearly $S = \bigcup_{s \in \mathbf{R}} (\{s\} \times F_s)$, where F_s is the set of fixed points of the mapping $K_s = K_{\{s, A\}}: V^\perp \rightarrow V^\perp$ defined by $K_s(w) = A[h - QG(s\phi + w)]$. From the compactness of A and the continuity and uniform boundedness of G it follows that K_s is compact and maps into the ball $B = B_\rho(0) = \{w \in V^\perp \mid \|w\|_H \leq \rho\}$, where

$$\rho = \|A\|(\|h\| + \sqrt{\pi} \sup |g(u)|) \quad (4.7)$$

Therefore, by Schauder's fixed point theorem, each F_s is nonempty, so that $\text{Proj}_\mathbf{R} S = \mathbf{R}$ and, in fact $S \subset \mathbf{R} \times B_\rho$. Now (4.4)–(4.5) is equivalent to solving the equation $\Phi(s, w) = t$ is S , where the mapping $\Phi: \mathbf{R} \times (D(L) \cap V^\perp) \rightarrow \mathbf{R}$ is given by

$$\Phi(s, w) = \int_0^1 g\left(x, s\phi + w, -\left(\frac{\pi}{2}\right)^2 \phi + w''\right) \phi dx \quad (4.8)$$

It is clear that Φ is continuous and bounded. This show the following

Lemma 4.2 Problem (1.9)–(1.10) is equivalent to solving the equation $\Phi(s, w) = t$ in S .

Lemma 4.3 Let (H1) and (H5) hold. Then, for each $s \in \mathbf{R}$, K_s has only one fixed point.

Proof From the definition of A and B , (4.4) is equivalent to the following system of equation

$$w = AQ \left[h - g \left(x, s\phi + w, - \left(\frac{\pi}{2} \right)^2 \phi + w'' \right) \right] \quad (4.9)$$

$$w'' = BQ \left[h - g \left(x, s\phi + w, - \left(\frac{\pi}{2} \right)^2 \phi + w'' \right) \right] \quad (4.10)$$

Suppose by contradiction that K has two distinct fixed points $w_1 \neq w_2$ in V^\perp , then

$$w_1 - w_2 = AQ \left[-g \left(x, s\phi + w_1, - \left(\frac{\pi}{2} \right)^2 \phi + w_1'' \right) + g \left(x, s\phi + w_2, - \left(\frac{\pi}{2} \right)^2 \phi + w_2'' \right) \right] \quad (4.11)$$

$$w_1'' - w_2'' = BQ \left[-g \left(x, s\phi + w_1, - \left(\frac{\pi}{2} \right)^2 \phi + w_1'' \right) + g \left(x, s\phi + w_2, - \left(\frac{\pi}{2} \right)^2 \phi + w_2'' \right) \right] \quad (4.12)$$

From (3.4), we obtain the result that

$$\|w_1 - w_2\| \leq \|A\| \{a\|w_1 - w_2\| + b\|w_1'' - w_2''\|\} \quad (4.13)$$

$$\|w_1'' - w_2''\| \leq \|B\| \{a\|w_1 - w_2\| + b\|w_1'' - w_2''\|\} \quad (4.14)$$

By combining (4.13)–(4.14) and using (3.3), we obtain that $\|w_1 - w_2\| = \|w_1'' - w_2''\| = 0$. Thus we arrive at the desired contradiction that $w_1 = w_2$.

A immediate consequence of Lemma 4.3 and Lemma 4.1 is the following

Corollary 4.4 Let (H1) and (H5) hold. Then S is connected.

Now, define W as the projection of S over V^\perp , i. e.,

$$W = \{w \mid (s, w) \in S \text{ for some } s \in \mathbf{R}\}.$$

Lemma 4.5 Let (H1) hold. Then W is a bounded set in $C^1[0, 1]$.

Proof For each $w \in W$, from the definition of W , we see that there exists $(s, w) \in S$ such that

$$w^{(4)} - \alpha_1 w + \beta_1 w'' = h - g \left(x, s\phi + w, - \left(\frac{\pi}{2} \right)^2 \phi + w'' \right) \quad (4.15)$$

(4.15) is equivalent to the following system of equations

$$w = AQ \left[h - g \left(x, s\phi + w, - \left(\frac{\pi}{2} \right)^2 \phi + w'' \right) \right] \quad (4.16)$$

$$w'' = BQ \left[h - g \left(x, s\phi + w, - \left(\frac{\pi}{2} \right)^2 \phi + w'' \right) \right] \quad (4.17)$$

Condition (3.1) together with the compactness of A and B imply that there exist C_1, C_2 , such that for all for all for all

$$\|w\|_B \leq C_1 \quad \forall w \in W \quad (4.18)$$

$$\|w''\|_B \leq C_2 \quad \forall w \in W \quad (4.19)$$

By combining (4.18) and (4.19) and (3.1) with (4.15), we obtain the existence of a constant $C_3 > 0$, such that

$$\|w^{(4)}\|_H \leq C_2 \quad \forall w \in W \quad (4.20)$$

Since $w \in D(L) \cap V^4$ so that

$$w(0) = w''(0) = w'(1) = w'''(1) = 0$$

and

$$w^{(4)}(x) = w^{(4)}(x) - w^{(4)}(1) = \int_0^x w^{(5)}(s) ds, \quad 0 < x < 1,$$

it follows easily from (4.20) that

$$\|w^{(4)}\|_{C([0,1])} \leq C_3 \quad (4.21)$$

By (4.21) and the fact

$$w''(x) = w''(x) - w''(0) = \int_0^x w^{(3)}(s) ds$$

we get that

$$\|w''\|_{C([0,1])} \leq C_3$$

Similarly,

$$\|w'\|_{C([0,1])} \leq C_2, \quad \|w\|_{C([0,1])} \leq C_2$$

Thus

$$\|w\|_{C^4([0,1])} \leq 2(C_3 + C_2).$$

Corollary 4.6 There exists $\beta > 0$ such that

$$\left. \begin{aligned} s\phi + w(x) &\geq 0, \quad -s\phi + w(x) \leq 0 \\ s\left(\frac{\pi}{2}\right)^2 \phi + w(x) &\geq 0, \quad -s\left(\frac{\pi}{2}\right)^2 \phi + w(x) \leq 0 \end{aligned} \right\} \quad (4.22)$$

for all $s \geq \beta$, $w \in W$, $x \in [0, 1]$.

Proof For every $w \in W$, let us define $\psi: [0, 2] \rightarrow \mathbf{R}$ by

$$\psi(x) = \begin{cases} w(x) & x \in [0, 1] \\ w(2-x) & x \in (1, 2] \end{cases} \quad (4.23)$$

Then $\psi \in C_b^4([0, 2])$. Now using the well-known estimate (see, e. g. estimate (16) of [6]), we get

$$|\psi(x)| \leq \max |\psi'(x)| \sin \frac{\pi x}{2} \leq \frac{C_2}{\sqrt{2}} \phi, \quad \forall x \in [0, 2] \quad (4.24)$$

Similarly

$$|w''(x)| \leq \frac{C_3}{\sqrt{2}} \phi \quad (4.25)$$

Corollary 4.7 Let (H1), (H2) and (H3) hold. Then there exists $\beta_1 > 0$ such that $\Phi(s, w) < 0$ and $\Phi(-s, w) \geq 0$ for all $s \geq \beta_1$ and $w \in W$.

Proof Let $\beta > 0$ be given by Corollary 4.6 and take β_0 so that $\beta_0 \max \phi - C > y_1$, where

$$C = \sup_{w \in W} |w|_{C([0,1])}.$$

Letting

$$\beta_1 = \beta + \beta_0,$$

we have

$$\begin{aligned} s\phi + w(x) &\geq \beta_0 \phi(x) \geq 0 \\ -s\phi + w(x) &\leq -\beta_0 \phi(x) \leq 0 \\ s\left(\frac{\pi}{2}\right)^2 \phi + w''(x) &\geq \beta_0 \left(\frac{\pi}{2}\right)^2 \phi(x) \geq 0 \\ -s\left(\frac{\pi}{2}\right)^2 \phi + w''(x) &\leq -\beta_0 \left(\frac{\pi}{2}\right)^2 \phi(x) \leq 0 \end{aligned}$$

for all $s \geq \beta_1$, $w \in W$ and $x \in [0, 1]$. Hence, by (H3) we get

$$\Phi(s, w) = \int_0^1 g\left(x, s\phi + w, -s\left(\frac{\pi}{2}\right)^2 \phi + w''\right) \phi dx \leq 0 \quad (4.26)$$

$$\Phi(-s, w) = \int_0^1 g\left(x, -s\phi + w, s\left(\frac{\pi}{2}\right)^2 \phi + w''\right) \phi dx \geq 0 \quad (4.27)$$

For all $s \geq \beta_1$, $w \in W$

In fact, we have strict inequality in (4.24). Since for each $s \geq \beta_1$ and $w \in W$, the function $s\phi(x) + w(x)$ is zero at $x=0$, and

$$s\sqrt{2} \sin \frac{\pi}{2} + w(1) \geq \beta_0 \max \phi - C > y_1$$

Therefore $s\phi(x_+) + w(x_+) = y_1$ for some $x_+ \in (0, 1)$ so that

$$\begin{aligned} &g\left(x_+, s\phi(x_+) + w(x_+), -s\left(\frac{\pi}{2}\right)^2 \phi(x_+) + w''(x_+)\right) \\ &= g\left(x_+, y_1, -s\left(\frac{\pi}{2}\right)^2 \phi(x_+) + w''(x_+)\right) < 0 \end{aligned}$$

Proof of Theorem 3.1 As we already observed in Lemma 4.2 (1.9)–(1.10) is equivalent to solving the equation

$$\Phi(s, w) = t$$

in S . So (1.9)–(1.10) has at least one solution if and only if $t \in \Phi(S)$.

Let $\tau_1 = \inf \Phi(S)$ and $\tau_2 = \sup \Phi(S)$. Since g is bounded, then $\Phi(S)$ has to be bounded. This implies that

$$-\infty < \tau_1 \leq \tau_2 < +\infty$$

By Corollary 4.4, $\Phi(S)$ is connected.

Next, Corollary 3.7 gives

$$\Phi(s, w) < 0 \quad \forall (s, w) \in S, s \geq \beta_1 \quad (4.28)$$

and

$$\Phi(s, w) \geq 0 \quad \forall (s, w) \in S, \quad s \leq -\beta_1 \quad (4.29)$$

for some $\beta_1 > 0$. From this we conclude that $0 \in \Phi(S)$ and $-\infty < \tau_1 < 0 \leq \tau_2 < +\infty$.

Now (H4) together with Corollary 4.6 imply that

$$\lim_{\substack{|s| \rightarrow \infty \\ (s, w) \in S}} \Phi(s, w) = 0 \quad (4.30)$$

From (4.30) and (4.29), we conclude that

$$\Phi(S) = [\tau_1, \tau_2] \quad (4.31)$$

Therefore all that we have to show is that (1.9)–(1.10) has at least two distinct solution if $t \in (\tau_1, \tau_2) \setminus \{0\}$. We shall show this for $t \in (\tau_1, 0)$, the proof on $(0, \tau_2)$ is analogous.

Let $t \in (\tau_1, 0)$ be given and let $\Phi(s_0, w_0) = \tau_1$, where $(s_0, w_0) \in S$. By combining (4.28) and (4.30) and Corollary 4.4 we conclude that there are two constants s_1, s_2 : $s_1 < s_0 < s_2$ such that

$$\Phi(s_i, w_{s_i}) = t, \quad \text{for } (i = 1, 2)$$

where $(s_i, w_{s_i}) \in S$. Thus $s_1\phi + w_{s_1}$ and $s_2\phi + w_{s_2}$ are two distinct solutions of (1.9)–(1.10).

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