Applied Mathematics and Mechanics
(English Edition, Vol. 16, No. 10, Oct. 1995)

Published by SU,
Shanghai, China

NECESSARY AND SUFFICIENT CONDITIONS FOR THE ABSOLUTE STABILITY OF DISCRETE TYPE LURIE CONTROL SYSTEM

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(Received Oct. 5, 1994; Communicated by Li Li)*

Abstract

In this paper, it is discussed that the absolute stability for zero solution of the discrete type Lurie control system

$$\left.\begin{aligned} x(n+1)&=Ax(n)+bf[\sigma(n)] \\ \sigma(n)&=c^T x(n) \end{aligned}\right\} \tag{1}$$

in which the nonlinear function $f(\sigma)$ satisfying conditions as follows

$$f(0)=0,\ \sigma f(\sigma)>0 \qquad (\sigma\neq 0) \tag{2}$$

or

$$f(0)=0,\ 0\leqslant k_1\leqslant f(\sigma)/\sigma\leqslant k_2<+\infty \qquad (\sigma\neq 0) \tag{3}$$

It gives the necessary and sufficient conditions for the absolute stability for system (1) under conditions (2). We also obtain the sufficient criteria for absolute stability of the simplified system of (1) under conditions (3).

Key Words discrete type, absolute stability, necessary and sufficient condition, Lurie problem

I. Introduction

More than forty years ago, the Pro-Soviet Union author, A. I. Lurie proposed the Lurie control system and Lurie problem[1], which have general significance in the nonlinear control theory and control engineering, by investigating the stability of automatic operating instrument of aircraft. Since that time, many authors have extensively studied Lurie control systems to describe in various forms and obtained a lot of results for absolute stability[2-8]. Unfortunately, it was only obtained the sufficient criteria for absolute stability of Lurie control system or necessary and sufficient conditions for some special classes[6-8]. Up to now, there is not a complete and constructive result for Lurie control systems to describe in various forms.

In Refs. [9-11], the absolute stability of zero solution is investigated for the discrete type Lurie control systems as follows

$$\left.\begin{aligned} x(n+1)&=Ax(n)+bf[\sigma(n)] \\ \sigma(n)&=c^T x(n) \end{aligned}\right\} \tag{1.1}$$

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*First Received July 5, 1994

where matrix $A=(a_{ij})_{m\times m}\in R^{m\times m}$, vectors $x,\ b,\ c\in R^m$, nonlinear function $f(\sigma)$ satisfying conditions as follows

$$f(0)=0,\quad \sigma f(\sigma)>0 \qquad (\sigma\neq 0) \tag{1.2}$$

or

$$f(0)=0,\quad 0\leqslant k_1\leqslant f(\sigma)/\sigma\leqslant k_2<+\infty, \qquad (\sigma\neq 0) \tag{1.3}$$

in [9]. Liao Xiaoxin obtained the necessary and sufficient conditions and some sufficient algebraical criteria for the absolute stability of zero solution of system (1.1) under conditions (1.2) (called infinite sector condition) or conditions (1.3) (called finite sector condition). The criteria are available under finite sector condition, but they are unavailabe under infinite condition except some special cases. So are results in [10, 11].

In this paper, firstly, we establish the dimension reducing principle. By using the principle, we obtain the explicitly necessary and sufficient condition for the absolute stability of system (1.1) under infinite sector condition. we also give some sufficient algebraical criteria for system (1.1) under finite sector condition.

II. Necessary and Sufficient Conditions for Absolute Stability of System (1.1) under Infinite Sector Condition

We establish the dimension reducing principle firstly. Similar to [12], we obtain the Lemma as follow

Lemma 1 If $\mathrm{Rank}[c,\ A^Tc,\cdots,(A^{m-1})^Tc]=m_1$, then there exists a nonsingular linear transformation $x=My$, such that system (1.1) can be transformed into

$$\left.\begin{aligned} &y(n+1)=\bar{A}y(n)+\bar{b}f(\sigma(n))\\ &\sigma(n)=\bar{c}^Ty(n)\end{aligned}\right\} \tag{2.1}$$

where

$$\bar{A}=\begin{bmatrix}\bar{A}_{11} & 0\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix},\quad \bar{b}=\begin{bmatrix}\bar{b}_1\\ \bar{b}_2\end{bmatrix},\quad \bar{c}=\begin{bmatrix}\bar{c}_1\\ 0\end{bmatrix},\quad y=\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$\bar{A}_{11}\in R^{m_1\times m_1}$, $\bar{A}_{22}\in R^{m_2\times m_2}$, $\bar{b}_1$, $\bar{c}_1$, $y_1\in R^{m_1}$, $\bar{b}_2$, $y_2\in R^{m_2}$, $m_1+m_2=m$, and

$$\mathrm{Rank}[\bar{c}_1,\ \bar{A}_{11}^T\bar{c}_1,\cdots,\ (\bar{A}_{11}^{m_1-1})^T\bar{c}_1]=m_1$$

Theorem 1 If A is Schur stable, namely spectral radius $R(A)<1$, then the absolute stability of zero solution of system (1.1) is equivalent to that of subsystem of (2.1)

$$\left.\begin{aligned} &y_1(n+1)=\bar{A}_{11}y_1(n)+\bar{b}_1f(\sigma(n))\\ &\sigma(n)=\bar{c}_1^Ty_1(n)\end{aligned}\right\} \tag{2.2}$$

Proof Because the transformation $x=My$ is nonsingular, the absolute stability of system (1.1) is equivalent to that of system (2.1). It is obvious that the absolute stability of zero solution of system (2.1) implies that of system (2.2). Similar to the proofs of [10, 11], it is easy to prove that if zero solution of system (2.2) is absolute stable, then zero solution of system (2.1) is absolute stable, that is, zero solution of system (1.1) is absolute stable. The proof is complete.

Lemma 1 and Theorem 1 construct the reducing dimension principle. It is obvious that the reducing dimension principle is true under finite sector condition. Contrasting with Ref. [11], the principle in this paper is explicit and more convenient for application

For the m_1 dimensional system (2.2), due to $\mathrm{Rank}[\bar{c}_1, \bar{A}_{11}^T\bar{c}_1, \cdots, (\bar{A}_{11}^{m_1-1})^T\bar{c}_1] = m_1$, there exists a nonsingular transformation[12], $z_1 = N^{-1}y_1$, such that system (2.2) can be changed into the form

$$\left.\begin{aligned} z_1(n+1) &= \alpha z_1(n) + \beta f(\sigma) \\ \sigma &= e^T z_1(n) \end{aligned}\right\} \tag{2.3}$$

in which

$$N^{-1} = \begin{bmatrix} 1 & a_{m_1-1} & \cdots & a_2 & a_1 \\ 0 & 1 & \cdots & a_3 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{m_1-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{c}_1^T \bar{A}_{11}^{m_1-1} \\ \bar{c}_1^T \bar{A}_{11}^{m_1-2} \\ \vdots \\ \bar{c}_1^T \bar{A}_{11} \\ \bar{c}_1^T \end{bmatrix}$$

$$\alpha = N^{-1}\bar{A}_{11}N = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -a_{m_1-1} \end{bmatrix}$$

$$\beta = N^{-1}\bar{b}_1 = (\beta_0, \beta_1, \beta_2, \cdots, \beta_{m_1-1})^T, \quad e = \bar{c}_1^T N = (0,0,0,\cdots,0,1)^T$$

$a_i (i = 0, 1, 2, \cdots, m_1 - 1)$ are the coefficients of the characteristic polynomial of the matrix $\bar{A}_{11}$, namely

$$\det(\lambda I - \bar{A}_{11}) = \lambda^{m_1} + a_{m_1-1}\lambda^{m_1-1} + \cdots + a_0$$

where I is a m_1-th order identity matrix.

Lemma 2 Under infinite sector condition, the necessary condition for absolute stability of systm (2.3) is

$$\beta_i = 0 \qquad (i = 0, 1, \cdots, m_1 - 1)$$

that is, the necessary condition for the absolute stability of the zero solution of system (2.2) is $\bar{b}_1 = 0$.

Proof For system (2.3), let $f(\sigma) = k\sigma$, in which $k > 0$ is arbitrary. Let

$$F(k) = \alpha + k\beta e^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 + k\beta_0 \\ 1 & 0 & \cdots & 0 & -a_1 + k\beta_1 \\ 0 & 1 & \cdots & 0 & -a_2 + k\beta_2 \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -a_{m_1-1} + k\beta_{m_1-1} \end{bmatrix}$$

If the zero solution of system (2.3) is absolute stable, for arbitrary given linear function $f(\sigma) = k\sigma (k > 0)$ satisfying condition (1.2), the zero solution of system (2.3) is globally asymptotically stable. Thus $F(k)$ is Schur stable for arbitrary given $k>0$.

The characteristic equation of $F(k)$ is

$$\det[\lambda I-F(k)]=\lambda^{m_1}+(a_{m_1-1}-k\beta_{m_1-1})\lambda^{m_1-1}+\cdots+(a_0-k\beta_0)=0 \qquad (2.4)$$

For Eq. (2.4), characteristic roots λ_i satisfy $|\lambda_i|<1\ (i=1,2,\cdots,m_1)$. According to the relation between the roots and the coefficients of Eq. (2.4), we have

$$\sum_{i=1}^{m_1}\lambda_i=-(a_{m_1-1}-k\beta_{m_1-1})$$

$$\sum_{\substack{i,j=1\\(i<j)}}^{m_1}\lambda_i\lambda_j=a_{m_1-2}-k\beta_{m_1-2}$$

$$\cdots$$

$$\prod_{i=1}^{m_1}\lambda_i=(-1)^{m_1}(a_0-k\beta_0)$$

Therefore $(\alpha_i-k\beta_i)$ $(i=0,\ 1,\ \cdots,\ m_i-1)$ are bounded. Because of arbitrarity of $\breve{k}$, we get $\beta_i=0$ $(i=0,\ 1,\ 2,\ \cdots,\ m_i-1)$. Forthmore we know $\bar{b}_1=0$ from the relation between system (2.2) and (2.3). The proof is complete.

From the proceeding in the proof of Lemma 2, we know that if the zero solution of system (1.1) is absolute stable, then by a nonsigülar transformation, system (1.1) can be changed into the form

$$\left.\begin{aligned}&\begin{bmatrix}y_1(n+1)\\y_2(n+1)\end{bmatrix}=\begin{bmatrix}\bar{A}_{11}&0\\\bar{A}_{21}&\bar{A}_{22}\end{bmatrix}\begin{bmatrix}y_1(n)\\y_2(n)\end{bmatrix}+\begin{bmatrix}0\\\bar{b}_2\end{bmatrix}f(\sigma)\\&\sigma=[\bar{c}_1^T,\ 0]\begin{bmatrix}y_1(n)\\y_2(n)\end{bmatrix}\end{aligned}\right\} \qquad (2.5)$$

Using the same method in the proof of Theorem 1, we can prove that if $\bar{A}_{11}$, $\bar{A}_{22}$ are Schur stable, the absolute stability of zero solution of systems (2.5) is equivalent to asymptotical stability of the zero solution of the linear system

$$y_1(n+1)=\bar{A}_{11}y_1(n) \qquad (2.6)$$

According to the above discussing, we get the result.

Theorem 2 If A is Schur stable, under infinite sector condition, the necessary and sufficient condition is that there exists a nonsigular transformation $x=Ty$, such that system (1.1) can be changed into system (2.5).

Now, we give a result to be convenient for application.

Theorem 3 If A is Schur stable, under infinite sector condition, the necessary and sufficient condition for absolute stability of zero solution of system (1.1) is

$$c^TA^ib=0 \qquad (i=0,1,2,\cdots,m-1)$$

Proof *Necessary* Because the zero solution of systems (1.1) is absolutely stable, so is that of systems (2.2) and (2.3). It implies $\beta_i=0\,(i=0,1,\cdots,m_1-1)$, i.e. $\beta=0$, $\bar{b}_1=N\beta=0$. According to the theory of matrix,

$$c^T A^i b = \bar{c}\bar{A}^i \bar{b} = \bar{c}_1^T \bar{A}_{11}^i \bar{b}_1 = 0 \qquad (i = 0, 1, 2, \cdots, m-1)$$

Sufficient The absolute stability of system (1.1) is equivalent to that of (2.1). Since

$$c^T A^i b = \bar{c}^T \bar{A}^i \bar{b} = \bar{c}_1^T \bar{A}_{11}^i \bar{b}_1 = 0 \qquad (i = 0, 1, \cdots, m_1 - 1)$$

$$[\bar{c}_1, \ \bar{A}_{11}^T \bar{c}_1, \ \cdots, \ (\bar{A}_{11}^{m_1-1})^T \bar{c}_1]^T \bar{b}_1 = 0$$

and

$$\mathrm{Rank}[\bar{c}_1, \ \bar{A}_{11}^T \bar{c}_1, \cdots, \ (\bar{A}_{11}^{m_1-1})^T \bar{c}_1] = m_1$$

namely,

$$\det[\bar{c}_1, \ \bar{A}_{11}^T \bar{c}_1, \ \cdots, \ (\bar{A}_{11}^{m_1-1})^T \bar{c}_1] \neq 0$$

we can get $\bar{b}_1 = 0$. So the system (2.1) has the form of (2.5). Due to Theorem 2, Theorem 3 is true. The proof is finished.

Because the system (1.1) has special structure when its zero solution is absolutely stable under infinite sector condition, it is difficult to obtain available criteria for absolute stability by constructing Liapunov function. In the procedure to prove above theorems, we also prove that the Aizerman conjecture [2] is true under this specific case, namely, the absolute stability of zero solution of (1.1) is equivalent to the globally asymptotical stability of zero solution of its linearized system $x(n+1) = (A + kbc^T)x(n)$ $(k > 0)$ being arbitrary. For continuous type Lurie control system [5], if $c^T b = 0$, $c^T A b = 0$, then the criteria in [5] is unavailable, which is pointed out by [14] when A being identity matrix and $c^T b = 0$, but the problem can be solved by using the mothod in this paper.

III. The Absolute Stability of Zero Solution of System (1.1) under Finite Sector Condition

Without loss of generality, suppose $b_m \neq 0$ (or, interchanging the situation of equation and variable). If $c^T b \neq 0$, we construct the transformation as follows

$$\begin{bmatrix} x^{(1)} \\ \sigma \end{bmatrix} \doteq Lx \tag{3.1}$$

where $x^{(1)} = \mathrm{col}(x_1^{(1)}, \ x_2^{(1)}, \cdots, \ x_{m_1-1}^{(1)})$

and

$$L = \begin{pmatrix} b_m & 0 & \cdots & 0 & -b_1 \\ 0 & b_m & \cdots & 0 & -b_2 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & b_m & -b_{m-1} \\ c_1 & c_2 & \cdots & c_{m-1} & c_m \end{pmatrix}$$

By the transformation (3.1), the system (1.1) can be changed into

$$\left.\begin{aligned} x^{(1)}(n+1) &= Bx^{(1)}(n) + h\sigma(n) \\ \sigma(n+1) &= g^T x^{(1)}(n) + p\sigma(n) - \rho f[\sigma(n)] \end{aligned}\right\} \tag{3.2}$$

in which

$$h = \mathrm{col}(h_1, \cdots, h_{m-1}), \quad g = \mathrm{col}(g_1, \cdots, g_{m-1})$$

$$\begin{bmatrix} B & h \\ g^T & p \end{bmatrix} = LAL^{-1} \triangleq A^{(1)}, \quad B = (b_{ij})_{(m-1)\times(m-1)}$$

System (3.2) is called the simplified system of system (1.1). The simplified system has many advantages: ① the transformation (3.1) is simple, ② there doesn't exist restriction for the system (1.1) except $c^Tb \neq 0$, ③ there don't exist extra variables and the coefficient column matrix of the nonlinear term is the simplest.

Similar to theorem 4 in [9], it is not difficult to obtain the result.

Theorem 4 If there exist constant numbers $t_i > 0 (i = 1, 2, \cdots, m)$ such that

$$\max_{1 < j < m-1} \left\{ \sum_{i=1}^{m-1} \frac{t_i}{t_j} |b_{ij}| + \frac{t_m}{t_j} |g_i| \leqslant 1 \right.$$

$$\sum_{j=1}^{m} \frac{t_j}{t_m} |h_j| + \max_{k=k_1,k_2} |p - \rho k| \leqslant \mu < 1$$

hold, then the zero solution of system (1.1) is absolutely stable.

Theorem 5 For system (3.2), if $b_{ij} \geqslant 0$, $h_i \geqslant 0$, $g_i \geqslant 0$, then the necessary and sufficient condition for absolute stability of zero solution of (3.2) is that leading principal minors of the matrix $I - A^{(2)}$ are positive.

Proof *Necessary* Let $f(\sigma) = k\sigma (k_1 \leqslant k \leqslant k_2)$, the linearized system of system (3.2)

$$\begin{bmatrix} x^{(1)}(n+1) \\ \sigma(n+1) \end{bmatrix} = \begin{bmatrix} B & h \\ g^T & p - \rho k \end{bmatrix} \begin{bmatrix} x^{(1)}(n) \\ \sigma(n) \end{bmatrix} \tag{3.3}$$

is Schur Stable. By Theorem 9.16 in [13], we know that the leading principal minors are positive.

Sufficient Due to the given conditions, there exists vector $t = \mathrm{col}(t_1, \cdots, t_m) (t_i > 0, i = 1, 2, \cdots, m)$ such that $A^{(2)} t < t$, namely, satisfying the conditions of Theorem 4, therefore the Theorem 5 is true.

I am grateful to professor Shu Zhongzhou for his disection, professor Hong Jingfeng for his helpful and Professor Li Li for many suggestions and useful discussions.

References

[1] A. M. Letov, Stability of Nonlinear Control Systems (1955). (in Russian), English tr. of 1st ed., Princeton University Press, Princeton (1961).

[2] Shu Zhongzhou, Stability of Motion, Press of Southwest Jiaotong University, Chengdu (1989). (in Chinese)

[3] Liao Xiaoxin, *Mathematics Theory and Application of the Stability*, Press of Central China Normal University, Wuhan (1988). (in Chinese)

[4] Qin Yuanxun, et al., *Theory and Application of stability of Motion*, Science Press, **Beijing** (1981). (in Chinese)

[5] Zhu Siming, On the criterion of absolute stability for system of direct control, Acta Scientiarum Naturalium University Sunyatseni, 3 (1979), 20—28. (in Chinese)

[6] Xie Huimin, Necessary and sufficient conditions for absolute stability of a class of third order control systems, *Scientia Sinica (Series A)* (English Ed.), 5 (1982), 527—540.

[7] Zhang Jiye and Shu Zhongzhou, Necessary and Sufficient criteria for absolute stability of the direct control system, *Appl. Math. and Mech.* (English Ed.), **15**, 3 (1994), 259–266.

[8] Qiu Xiaogang and Shu Zhongzhou, Several criteria of absolute stability for the second canonical form of control system, *Appl. Math. and Mech.* (English Ed.), **7** , 9 (1986), 841–856.

[9] Liao Xiaoxin, The necessary and sufficient conditions for absolute stability of discrete type Lurie control system, *Chinese Annals of Mathematics*, **10A**, 5 (1989), 628–635. (in Chinese)

[10] Lj. T. Grujic and D. D. Siljak, Exponential stability of Large-scale direct systems, *Int. J. Control.*, **19**, 3 (1974), 481–491.

[11] Xiao Shuxian, The dimension reducing principle and application of discrete type Lurie control system, *J. Huazhong Univ. of Sci. and Tech.*, **20** (sup) (1992), 97–101. (in Chinese)

[12] Chen Chitsong, *Linear System Theory and Design*, 2nd ed. Holt, Rinehart and Winston, New York (1984).

[13] J. P. La Sall., *The Stability of Dynamical Systems*, SIAM, Philadelphia, (1976).

[14] Li Senlin, The necessary and sufficient conditions for the absolute stability of several classes of direct control systems, *Acta Mathematicae Applicate Sinica*, **6**, 4 (1983), 458–467. (in Chinese)