

TURBULENT COHERENT STRUCTURE AND DYNAMIC SYSTEM*

Li Jia-chun (李家春)

(LNM, Institute of Mechanics, CAS, Beijing)

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Abstract

The main difficulties in the study of turbulence via dynamic system lie in how to relate continuum systems of infinite dimension with dynamic system in low dimension space and how to depict its spacial structure. In this paper, we'll give a comprehensive review on various methods to describe complex systems in low dimension space and new approaches to the resolution of turbulence problems.

Key words turbulence, coherent structure, dynamic system, low dimension formulation, Karhunen-Loeve expansion

Turbulence is a most crucial and tough problem in fluid mechanics. In the past hundred years since Reynolds pioneer work, people have made good progress in this respect, though it still constitutes great challenges to applied mathematicians and fluid dynamicists whether breakthrough in the area can be made in this century. Since 1970s', people have succeeded in describing turbulence by dynamic system. On the other hand, they have also found coherent structures in shear turbulence experiments. Both of them are advancing simultaneously. Recently, people found the relations between them, which seems to shed light on the resolution of the problem.

I. Low Dimension Formulation of Dynamic System

Generally speaking, people always try to describe a complicated physical phenomenon with as few variables as possible, or in as low dimension space as possible so as to save much effort and study from more intuitive geometric point of view. By the end of last century, Poincaré proposed phase plane concept. What Soviet expert Andronov was engaged in for all his life was to investigate the qualitative behaviour of integral orbits in the phase plane. Decades of arduous efforts have been made from two-dimension system to three-dimension one due to their radically different properties. Moreover, many mathematical conclusions can not be deduced from finite dimension space to infinite one. For this reason, Functional was invented to answer those problems in infinite dimension space. The correspondence between dynamic system in finite dimension space and turbulence is the very problem we want to study in details. Now, let's look back on the methods for reducing dimension number available at first: direct sum decomposition; centre manifold theorem; Liapunov-Schmidt procedure.

For linear dynamic system in the space of n dimension

$$d\mathbf{X}/dt = A\mathbf{X}, \quad \mathbf{X}|_{t=0} = \mathbf{X}_0 \quad (1.1)$$

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where A is $n \times n$ matrix, the symbol “—” indicates vectors. The solution of (1.1) can be represented in the form of matrix function:

$$\underline{X} = \underline{X}_0 \exp[At] \quad (1.2)$$

Since matrix A must be similar to a Jordan's canonical one, i.e. there exists a matrix S with full rank so that

$$S^{-1}AS = \begin{pmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & \ddots \\ & & & J_r \end{pmatrix} \quad (1.3)$$

is satisfied, where J_i ($i = 1, 2, 3, \dots, r$) are Jordan's submatrix.

$$\exp[At] = S^{-1}(\exp[J_1 t] \oplus \exp[J_2 t] \oplus \dots \oplus \exp[J_r t])S \quad (1.4)$$

where \oplus is direct sum. Thus, the behaviour of solution flows can be analyzed in lower dimension space by direct sum decomposition.

As for nonlinear system, we can linearize the system nearby the equilibrium points. If it is hyperbolic, then Hartman-Grobman theorem assures that the system is structurally stable. That is to say, nonlinear flow and linear one are homeomorphism and the stable and unstable manifolds of two systems are tangent to each other. To understand the qualitative behaviours of the flow nearby hyperbolic points, the centre manifold theorem can be made use of to reduce the dimension.

Center Manifold Theorem: Let f be a C^r vector field on R^n vanishing at the origin ($f(0) = 0$) and let $A = Df(0)$. Divide the spectrum of A into three parts, σ_s , σ_c , σ_u with

$$\operatorname{Re} \lambda \begin{cases} < 0 & \lambda \in \sigma_s \\ = 0 & \lambda \in \sigma_c \\ > 0 & \lambda \in \sigma_u \end{cases} \quad (1.5)$$

Let the eigenspaces of σ_s , σ_c , σ_u be E_s , E_c , and E_u , respectively. Then there exist C^r stable and unstable invariant manifolds W^u and W^s tangent to E^u and E^s at O and a C^{r-1} centre manifold W^c tangent to E^c at O . The stable and unstable manifolds are unique, but W^c need not be.

According to the above theorem, we are able to investigate the local bifurcation of nonlinear equations on the centre manifold so that the problems are simplified a great deal.

Finally, Liapunov-Schmidt procedure is adopted to diminish dimension in the cases of Banach Space. For example, we deal with nonlinear eigenvalue problem:

$$Au - \lambda u = 0 \quad (1.6)$$

where A is an operator $B \rightarrow B$, $A'(0) = 0$, its bifurcation point must be the eigenvalue of the equation:

$$Lu - \lambda u = 0 \quad (1.7)$$

where $L = A'(0)$. The eigenfunction span $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$ constitutes the zero space of the operator $L - \lambda^0 I$. Eq. (1.6) is equivalent to

$$Lu - \lambda^0 u = \delta u - Ru \quad (1.8)$$

where $\delta = \lambda - \lambda^0$, $R = A - L$. If the solution is separated into the direct sum of two parts

$$u = \sum_{j=1}^k C_j \phi_j + w \quad (1.9)$$

In accordance with solvability condition

$$\left\langle (\delta I - R) \left(w + \sum_{j=1}^k C_j \phi_j \right), \phi_i^* \right\rangle = 0 \quad (i=1, 2, \dots, k) \quad (1.10)$$

$$w = T(\delta I - R) \left(\sum_{j=1}^k C_j \phi_j + w \right) \quad (1.11)$$

where T is the pseudoadversive operator. Substituting eq. (1.11) into (1.10), i.e. $w = w(\delta, C_1, C_2, \dots, C_k)$, we obtain bifurcation equation reducing a problem with infinite dimension into that with finite one.

II. Turbulent Coherent Structure

In 1967, Kline et al. found that the sublayer in turbulent boundary layers is full of streamwise eddies. Brown Roshko revealed in 1971 that the mixed layer is by no means merely a turbulent wedge without any structure. Their results were rather heuristic in encouraging people to explore new approaches to turbulence.

The so-called "Turbulent Coherent Structure" is a mass of turbulent eddies with their phase correlated. It is also a recognizable and long-term sustained flow pattern. Their importance is that they play a leading role in the transport of mass, momentum and energy and exert significant effects on turbulent mixing, entrainment and noise, at least in the period of transition. People further suggested grooving and ribbing to control the development of turbulence.

As compared with Reynolds average methods, they include more information about their phase. If we define assembly average as follows:

$$\langle f(x, y, z, \phi) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f_i(x, y, z, \phi + \phi_i(t)) \quad (2.1)$$

Evidently, it is different from time average:

$$\bar{f}(x, y, z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(x, y, z, t') dt' \quad (2.2)$$

If physical variables are separated into sums of average, coherent and stochastic parts:

$$f(x, y, z, t) = F(x, y, z) + f_c(x, y, z, t) + f_r(x, y, z, t) \quad (2.3)$$

then

$$\bar{f} = F, \quad \langle f \rangle = F + f_c \quad (2.4)$$

Therefore, the flow patterns are still recognizable even after assembly averaging.

As compared with statistical theory of turbulence, coherent structure implies deterministic behaviour in random one. It is order in disorder. It also means that there exists a scale, only less than

which random behaviour becomes dominant. The dissipative small scaled motion plays a recessive role in the dynamic process. i.e. the balance of turbulent is only a constraint, but not driving force for turbulent transport.

Typical turbulent are: turbulent plug in pipes, hairpin vortex in boundary layers, turbulent spots, Taylor vortex between two rotating cylinders, Kármán vortices, vortex pairs in mixing layers and turbulent ring in jets. It can be imagined that shear turbulence consists of these elements with quasideterministic patterns. It seems that the study of them is simpler than time average method and statistical theory. Discrete vortex method and large eddy simulation are physically based on the above concept.

III. A New Approach to the Study of Turbulence

In order to explore turbulence via low dimension system, we have to answer the following questions in advance: whether turbulence is capable of being formulated in low dimension space? how can we describe it more effectively? What is the criterion for us to estimate the degree of approximation?

Let's look at the facts in the experiment: a series of bifurcations were observed in the viscous flow around a sphere

$Re = 36$	vortex shedding starts, $f_1 = 590$ Hz;
$Re = 54$	the second frequency appears;
$Re = 66 - 71$	chaotic (Ruelle-Takens-Newhouse Mode);
$Re = 76$	quasiperiodic motion with 3 frequencies;
$Re = 81 - 88$	chaotic;
$Re = 90$	quasiperiodic motion with 4 frequencies;
$Re = 140 - 143$	chaotic;

from which people found that there is similarity between fluid flow and low dimension dynamic system, i.e. ordered and chaotic motions emerge alternatively. Moreover, people did have observed quasi-periodic motions with 3 or 4 frequencies, a little similar to Landau conjecture. Ruelle-Takens theory does not exhaust all cases. Besides, we found that the frequency of vortex shedding usually varies continuously with Re and discontinuously when going through chaotic windows. In addition, dimensions are calculated based on experimental data. It equals dependent frequencies and jumps to a fraction in chaotic window. For example, $Re = 66 - 71$, $\nu = 4.4$; $Re = 81 - 88$, $\nu = 4.8$, when Reynolds number goes up to 10^4 , the dimension will not exceed 20. The fact supports the argument that fluid flow systems can be formulated by low dimension dynamic system.

The principle of pattern recognition implies much to us. Although a picture can be described by the information at each pixel or by expanding them into Fourier series, the procedure is by no means the most effective. As a matter of fact, a great many pictures can be stored in the brain and the recognition is instant. It means that men can make swift judgement by a little information. Take recognition of people as an example. It is because many average faces are stored in the brain beforehand: European, Asian, African. Chinese are further divided into southerner and northerner. Having all the information in brain, we'll recognize individuals by a few features of them on their cheeks, eyes, lips etc.. Hence, a picture can be described in low dimension space without difficulties. Of course, the formulation must be carried out in optimally selected Hilbert space.

Consider the following equation

$$\partial \underline{u} / \partial t = G(\underline{u}, R) \quad (3.1)$$

where R is a parameter (such as Re), \underline{u} has to satisfy some additional conditions (for instance, $\nabla \cdot \underline{u} = 0$, boundary condition etc.). Choosing orthogonal functions $\{\underline{\psi}^{(n)}\}$ satisfying the above additional condition. Obviously, they can be transformed into a set of new orthogonal ones by optimization. Thus

$$\underline{u} = \sum A_n \underline{\Phi}^{(n)} \quad (3.2)$$

Define the correlation

$$K_{\mu\nu}(\underline{x}, \underline{y}) = \langle \underline{u}_\mu(\underline{x}), \underline{u}_\nu^*(\underline{y}) \rangle = \sum_k \lambda_k \underline{\Phi}_\mu^{(k)}(\underline{x}) \underline{\Phi}_\nu^{(k)*}(\underline{y}) \quad (3.3)$$

i.e. we require $\langle A_n, A_m^* \rangle = \lambda_n \delta_{nm}$, then

$$K \underline{\Phi}^{(n)} = \lambda_n \underline{\Phi}^{(n)} \quad (3.4)$$

It follows that the eigenfunctions of correlation can be selected. The approximate expression looks like

$$\underline{u} \sim \underline{u}_N = \sum_1^N A_n \underline{\Phi}^{(n)} \quad (3.5)$$

By Galerkin method

$$\left(\underline{\Phi}^{(j)}, \frac{\partial}{\partial t} \underline{u}_N - G(\underline{u}_N, R) \right) = 0 \quad (j=1, 2, \dots, N) \quad (3.6)$$

we derive low dimension dynamic system. The most energy should be included in the system to diminish errors, i.e. the ratio of energy

$$E = \sum_1^N \lambda_n / \sum_1^\infty \lambda_n \approx 1 \quad (3.7)$$

has to approach to 1 (for example, more than 95%). In doing so, we are able to determine the truncated term-number to solve discrepancy that the solutions of different modes are inconsistent

The expansion, known as Karhunen-Loeve expansion, comes from pattern recognition. The advantages of the expansion are its spacial structure and the optimization in the minimum truncation terms.

IV. Illustrated Examples

1. Ginzburg-Landau equation^[11]

$$i \partial A / \partial t + q^2 (1 - i c_0) \partial^2 A / \partial x^2 - i \rho A + (1 + i \rho) |A|^2 A = 0 \quad (4.1)$$

with additional condition $A(x + 2\pi, t) = A(x, t)$, where c_0, ρ is real parameters, which stems from many physical problems, and the cubic Schrödinger equation is merely one of its special cases. Consider the situation for q ranging from 0.6 to 1.3. The correlation is based on the chaotic solution for $q=0.95$ with Liapunov exponent being 3.05. The first three eigenvalues are 0.5328, 0.0855 and 0.0013 and the energy of three modes corresponding to them amounts to 99.9% of total

one due to $\lambda^k < 10^{-k}$. The exact and approximate solutions are in good agreement.

2. Heat convection^[11]: Take $Pr = 0.72$, $Ra = 46000$. The correlation and hundreds of eigenvalues and eigenfunctions are obtained with the help of spectral method. It seems that 106 terms will occupy 80% of total energy, 285 terms 90% and 596 terms 95%. Even the imperfect results save 99% storage as compared with flow records (32)³ 4.

3. Pattern recognition. By diminishing errors and normalization of 115 specimen pictures, we are able to obtain the so-called eigenpictures. Generally speaking, the satisfying requirements can be met with less than 100 eigenpictures, i.e. the dimension 2^{14} of the system has been reduced to 100. The error turns out 7.8% for full face and less for localized one. Strange enough, the male eigenpictures can also be used for female only with error 3.9% for 40 terms. The example shows that the method is excellent in applications.

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