

ON A GENERALIZATION OF BERTRAND'S THEOREM*

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Abstract

Bertrand's theorem for the determination of the applied forces to a holonomic system from one of its first integrals, is extended to nonholonomic systems. Some interesting applications of this new result are also given.

Key words analytical mechanics, nonholonomic system, first integral, inverse problem of dynamics

I. Introduction

In Whittaker's classical book^[1], the following Bertrand's theorem obtained in 1852 was mentioned: given a first integral of a system, the applied forces to it can be determined, if the forces are assumed to depend on the coordinates but not on the velocities. This is an important property about the integral of dynamic system and is also an important inverse problem of dynamics. It was pointed out in [1]: "However, this integral can not be chosen at random, but must satisfy certain conditions." Nevertheless, this method using simple information to study the inverse problem of dynamics is rather delicate. Recently, Galiullin, in [2], introduced Yerugin function^[3] to this field and extended the above mentioned Bertrand's theorem. However, those approaches are only restricted to the holonomic systems.

We are to extend Bertrand's theorem to nonholonomic systems. As a result, we find an important property about the integral of nonholonomic systems, and solve a inverse problem of dynamics of nonholonomic systems. The idea is as follows.

First, we express the explicit form of the equations of nonholonomic systems with multipliers and treat it as a holonomic system with the condition of restrictions. Then we get a system of ordinary differential equations of second order.

Secondly, we differentiate the given integral with respect to time t . By introducing Yerugin function, we get an ordinary differential equation of second order. Combining with the above system of ordinary differential equations, we eliminate the generalized accelerations, and get an equation involving the generalized forces and about coordinates, velocities and time.

Finally, by this equation, we get a system of algebraic equations, from which the generalized forces can be determined.

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In this paper, two examples are given to show the concrete applications of this new result.

II. Explicit Form of the Equations of Motion of a Nonholonomic System

Suppose that the configuration of the given system is determined by n generalized coordinates $q_s (s=1, 2, \dots, n)$. There are g ideal nonholonomic constraints of Chetaev's type

$$f_\beta(q_s, \dot{q}_s, t) = 0, \quad (\beta=1, 2, \dots, g, s=1, 2, \dots, n) \quad (2.1)$$

The equations of motion can be expressed in the form of Routh equation^[4],

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s + \sum_{\beta=1}^g \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s=1, 2, \dots, n) \quad (2.2)$$

where T is the kinetic energy, Q_s are the generalized forces and λ_β are the undetermined multipliers. We may write (2.2) in an explicit form^[4],

$$\begin{aligned} & \sum_{k=1}^n A_{sk} \dot{q}_k + \sum_{k=1}^n \sum_{m=1}^n [k, m, s] \dot{q}_k \dot{q}_m \\ &= \sum_{k=1}^n \left(\frac{\partial B_s}{\partial q_k} - \frac{\partial B_k}{\partial q_s} \right) \dot{q}_k + Q_s - \frac{\partial B_s}{\partial t} + \frac{\partial T_0}{\partial q_s} - \sum_{k=1}^n \frac{\partial A_{ks}}{\partial t} \dot{q}_k + \sum_{\beta=1}^g \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \end{aligned} \quad (s=1, 2, \dots, n) \quad (2.3)$$

where

$$A_{sk} = A_{ks} = \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}, \quad B_s = \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \frac{\partial \mathbf{r}_i}{\partial t} \quad \text{and} \quad T_0 = \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial t}$$

m_i is the mass of the i -th particle in the system and \mathbf{r}_i is its radius vector. N is the total number of particles in the system. $[k, m, s]$ is defined as

$$[k, m, s] = \frac{1}{2} \left(\frac{\partial A_{ks}}{\partial q_m} + \frac{\partial A_{ms}}{\partial q_k} - \frac{\partial A_{km}}{\partial q_s} \right) \quad (2.4)$$

to be the Christoffel symbols of first class of A_{ks} . We get from (2.3),

$$\begin{aligned} \ddot{q}_l = \frac{\Delta_{sl}}{\Delta} \left\{ - \sum_{k=1}^n \sum_{m=1}^n [k, m, s] \dot{q}_k \dot{q}_m + \sum_{k=1}^n \left(\frac{\partial B_k}{\partial q_s} - \frac{\partial B_s}{\partial q_k} \right) \dot{q}_k + Q_s \right. \\ \left. - \frac{\partial B_s}{\partial t} + \frac{\partial T_0}{\partial q_s} - \sum_{k=1}^n \frac{\partial A_{ks}}{\partial t} \dot{q}_k + \sum_{\beta=1}^g \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right\}, \quad (l=1, 2, \dots, n) \end{aligned} \quad (2.5)$$

where $\Delta = |A_{sk}| \neq 0$ and Δ_{sl} is the algebraic complement minor of the element (s, l) in Δ .

Differentiating (2.1) with respect to t , we have

$$\sum_{i=1}^n \left(\frac{\partial f_r}{\partial q_i} \dot{q}_i + \frac{\partial f_r}{\partial \dot{q}_i} \ddot{q}_i \right) \frac{\partial f_r}{\partial t} = 0 \quad (r=1, 2, \dots, g) \quad (2.6)$$

substituting (2.5) into (2.6), we get the algebraic equations to determine the multipliers λ_β ,

$$\begin{aligned}
& \sum_{r=1}^g \sum_{s=1}^n \sum_{i=1}^n \frac{\Delta_{si}}{\Delta} \frac{\partial f_r}{\partial \dot{q}_i} \frac{\partial f_r}{\partial \dot{q}_s} \lambda_r + \sum_{i=1}^n \frac{\partial f_r}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial f_r}{\partial t} \\
& + \sum_{i=1}^n \frac{\partial f_r}{\partial \dot{q}_i} \sum_{s=1}^n \frac{\Delta_{si}}{\Delta} \left\{ - \sum_{m=1}^n \sum_{b=1}^n [k, m, s] \dot{q}_b \dot{q}_m + \sum_{b=1}^n \left(\frac{\partial B_b}{\partial \dot{q}_s} - \frac{\partial B_s}{\partial \dot{q}_b} \right) \dot{q}_b \right. \\
& \left. + Q_s + \frac{\partial T_0}{\partial \dot{q}_s} - \frac{\partial B_s}{\partial t} - \sum_{b=1}^n \frac{\partial A_{bs}}{\partial t} \dot{q}_b \right\} = 0 \quad (r=1, 2, \dots, g) \quad (2.7)
\end{aligned}$$

Since (2.1) satisfies Chetaev's condition, the rank of the coefficient matrix of $\lambda_r, (\partial f_r / \partial \dot{q}_i)$ is g . Then by (2.7), we have

$$\lambda_r = \sum_{s=1}^n a_{rs}(q, \dot{q}, t) Q_s + b_r(q, \dot{q}, t) \quad (2.8)$$

Substituting (2.5) into (2.8) and eliminating the multipliers λ_r , we get n ordinary differential equations, which are called the equations of motion of the corresponding holonomic system. If the initial conditions satisfy the nonholonomic constraints (2.1), the solution of the corresponding holonomic system is the solution of the original nonholonomic system.

III. Determination of the Generalized Forces

Given an integral of the nonholonomic system

$$\omega(q_s, \dot{q}_s, t) = C \quad (s=1, 2, \dots, n) \quad (3.1)$$

where ω is continuously differentiable with respect to all its variables. If C is an arbitrary constant, (3.1) is the first integral of the system; if C is a specified constant, (3.1) is a special integral of the system. Suppose that the generalized forces do not depend on the generalized velocities. We can determine the generalized forces by (3.1).

Differentiating (3.1) with respect to t and introducing Yerugin function, we have

$$\sum_{s=1}^n \left(\frac{\partial \omega}{\partial \dot{q}_s} \dot{q}_s + \frac{\partial \omega}{\partial \dot{q}_s} \dot{q}_s \right) + \frac{\partial \omega}{\partial t} = \Phi(q, \dot{q}, t) \quad (3.2)$$

where Φ is called Yerugin function. When C is an arbitrary constant,

$$\Phi = 0 \quad (3.3)$$

When C is a specified constant, Φ is an arbitrary function satisfying

$$\Phi|_{s=0} = 0 \quad (3.4)$$

Introducing Yerugin function is the key for studying the inverse problem of dynamics, especially for establishing stable system and constructing programming movement system^[2].

Substituting (2.5), (2.8) into (3.2) and eliminating \dot{q}_i , we have

$$\sum_{s=1}^n \sum_{i=1}^n \frac{\Delta_{si}}{\Delta} \frac{\partial \omega}{\partial \dot{q}_i} Q_s + \sum_{i=1}^n \frac{\partial \omega}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial \omega}{\partial t} - \Phi$$

$$\begin{aligned}
& + \sum_{i=1}^n \frac{\partial \omega}{\partial \dot{q}_i} \sum_{s=1}^n \frac{\Delta_{si}}{\Delta} \left\{ - \sum_{m=1}^n \sum_{k=1}^n [k, m, s] \dot{q}_k \dot{q}_m + \sum_{k=1}^n \left(\frac{\partial B_k}{\partial q_s} - \frac{\partial B_s}{\partial q_k} \right) \dot{q}_k \right. \\
& + \frac{\partial T_0}{\partial q_s} - \frac{\partial B_s}{\partial t} - \sum_{k=1}^n \frac{\partial A_{ks}}{\partial t} \dot{q}_k + \sum_{\beta=1}^g \left[\sum_{k=1}^n a_{\beta k}(q, \dot{q}, t) Q_k \right. \\
& \left. \left. + b_{\beta}(q, \dot{q}, t) \right] \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right\} = 0
\end{aligned} \quad (3.5)$$

We write (3.5) simply as

$$\sum_{s=1}^n a_s(q, \dot{q}, t) Q_s + b(q, \dot{q}, t) - \Phi = 0 \quad (3.6)$$

where

$$\begin{aligned}
a_s &= \sum_{i=1}^n \frac{\Delta_{si}}{\Delta} \frac{\partial \omega}{\partial \dot{q}_i} + \sum_{i=1}^n \frac{\partial \omega}{\partial \dot{q}_i} \sum_{k=1}^n \frac{\Delta_{ki}}{\Delta} \sum_{\beta=1}^g a_{\beta s} \frac{\partial f_{\beta}}{\partial \dot{q}_k}, \\
b &= \sum_{i=1}^n \frac{\partial \omega}{\partial q_i} \dot{q}_i + \frac{\partial \omega}{\partial t} + \sum_{i=1}^n \frac{\partial \omega}{\partial \dot{q}_i} \sum_{s=1}^n \frac{\Delta_{si}}{\Delta} \left\{ - \sum_{m=1}^n \sum_{k=1}^n [k, m, s] \dot{q}_k \dot{q}_m \right. \\
& \left. + \sum_{k=1}^n \left(\frac{\partial B_k}{\partial q_s} - \frac{\partial B_s}{\partial q_k} \right) \dot{q}_k + \frac{\partial T_0}{\partial q_s} - \frac{\partial B_s}{\partial t} - \sum_{k=1}^n \frac{\partial A_{ks}}{\partial t} \dot{q}_k + \sum_{\beta=1}^g b_{\beta} \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right\}. \quad (3.7)
\end{aligned}$$

The equation (3.6) consists of only q , \dot{q} and t , which is an identity for any independent variables appearing in it. So the partial derivatives of (3.6) with respect to \dot{q}_k is zero. By our assumption, Q_s do not depend on the generalized velocities. Therefore, we have

$$\sum_{s=1}^n \frac{\partial a_s}{\partial \dot{q}_k} Q_s + \frac{\partial b}{\partial \dot{q}_k} - \frac{\partial \Phi}{\partial \dot{q}_k} = 0 \quad (k=1, 2, \dots, n) \quad (3.8)$$

The equations (3.8) are the n -algebraic equations that we find to determine Q_s . The solutions of those equations provide an approach to determine the generalized forces Q_s . If the given integral is a first integral, i.e., C is an arbitrary constants, then we deduce from (3.8) the following equations

$$\sum_{s=1}^n \frac{\partial a_s}{\partial \dot{q}_k} Q_s + \frac{\partial b}{\partial \dot{q}_k} = 0 \quad (k=1, 2, \dots, n) \quad (3.9)$$

To sum up, we have the following theorem

Theorem Given the structure (2.1) and (2.2) of the nonholonomic system and assume that the generalized forces do not depend on the velocities, if the given integral (3.1) is a first integral, the generalized forces Q_s can be determined by (3.9); if the given integral (3.1) is a special integral, the generalized forces Q_s are determined by (3.8)

According to this theorem, it is possible to determine the generalized forces of a nonholonomic system, if one of the integrands of the system is given (whether it is a first integral or a special integral). This theorem may be said to be the generalized Bertrand's theorem for nonholonomic system.

The generalized Bertrand's theorem, on one hand, reveals an important property of nonholonomic systems, on the other hand, it provides an important and easy method to solve the inverse problems of dynamics of nonholonomic systems. If there is not any nonholonomic constraint to the system and the given integral is a first integral, the generalized Bertrand's theorem is just the original Bertrand's theorem; if there is not any nonholonomic constraint to the system and the given integral is a special integral, the generalized Bertrand's theorem is the theorem of Galiullin^[2].

It should be noted that all the Q_s 's can not be uniquely determined by (3.8) or (3.9). The reason are: (1) there appears the derivatives $\partial\Phi/\partial\dot{q}_k$ in (3.8). If the given integral is a special integral, Φ is arbitrary except that it satisfies (3.4). (2) If the given integral does not depend on all the \dot{q}_k ($k=1,2,\dots,n$) the number of equations in (3.8) or (3.9) is less than n . In these cases, to determine the generalized forces, we need some other supplementary conditions, e. g., condition on stability or on optimization.

IV. Examples

We give two examples to show the concrete applications of the result.

Example 1 Suppose that the configuration of a nonholonomic system is determined by three coordinates q_1 , q_2 and q_3 . The kinetic energy of it is

$$T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \quad (4.1)$$

Its movement is subject to a nonlinear nonholonomic constraint,

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \frac{a^2}{b^2}\dot{q}_3^2 = 0 \quad (4.2)$$

where a and b are constants. Given a first integral of the system,

$$\omega = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + mgq_3 = h \quad (4.3)$$

where g is the gravitational acceleration and h is an arbitrary constant. We use the generalized Bertrand's theorem to determine the generalized forces Q_1 , Q_2 and Q_3 . We assume that they depend only on coordinates.

First, we find the undetermined multiplier λ by (2.7). From the expression (4.1) of the kinetic energy

$$A_{ks} = \begin{cases} m & k=s \\ 0 & k \neq s \end{cases} \quad (k, s=1, 2, 3).$$

Then

$$\Delta_{sl} = \begin{cases} \frac{1}{m} & s=l \\ 0 & s \neq l \end{cases} \quad (s, l=1, 2, 3), \quad (4.4)$$

$$[k, m; s] = 0, \quad (4.5)$$

and

$$R_s=0, \quad T_0=0, \quad \frac{\partial A_{ss}}{\partial t}=0 \quad (4.6)$$

By the constraint equation (4.2), we have

$$\left. \begin{aligned} \frac{\partial f}{\partial \dot{q}_1} &= 2\dot{q}_1, & \frac{\partial f}{\partial \dot{q}_2} &= 2\dot{q}_2, & \frac{\partial f}{\partial \dot{q}_3} &= -2\frac{a^2}{b^2}\dot{q}_3 \\ \frac{\partial f}{\partial q_s} &= 0 \quad (s=1,2,3), & \frac{\partial f}{\partial t} &= 0 \end{aligned} \right\} \quad (4.7)$$

Substituting (4.4) - (4.7) into (2.7), we have

$$\frac{\lambda}{m} \left[(2\dot{q}_1)^2 + (2\dot{q}_2)^2 + \left(-2\frac{a^2}{b^2}\dot{q}_3 \right)^2 \right] + \frac{1}{m} \left[2\dot{q}_1 Q_1 + 2\dot{q}_2 Q_2 - 2\frac{a^2}{b^2}\dot{q}_3 Q_3 \right] = 0$$

And we get

$$\lambda = \frac{-\dot{q}_1 Q_1 - \dot{q}_2 Q_2 + a^2 b^{-2} \dot{q}_3 Q_3}{2(\dot{q}_1^2 + \dot{q}_2^2 + a^4 b^{-4} \dot{q}_3^2)} \quad (4.8)$$

Then we can deduce an algebraic equation like (3.6). The Routh equation now has the following form

$$m\ddot{q}_1 = Q_1 + 2\lambda\dot{q}_1, \quad m\ddot{q}_2 = Q_2 + 2\lambda\dot{q}_2, \quad m\ddot{q}_3 = Q_3 - 2\lambda\frac{a^2}{b^2}\dot{q}_3 \quad (4.9)$$

substituting (4.8) into (4.9), we have

$$\left. \begin{aligned} m\ddot{q}_1 &= Q_1 + \frac{\dot{q}_1(-\dot{q}_1 Q_1 - \dot{q}_2 Q_2 + a^2 b^{-2} \dot{q}_3 Q_3)}{\dot{q}_1^2 + \dot{q}_2^2 + a^4 b^{-4} \dot{q}_3^2} \\ m\ddot{q}_2 &= Q_2 + \frac{\dot{q}_2(-\dot{q}_1 Q_1 - \dot{q}_2 Q_2 + a^2 b^{-2} \dot{q}_3 Q_3)}{\dot{q}_1^2 + \dot{q}_2^2 + a^4 b^{-4} \dot{q}_3^2} \\ m\ddot{q}_3 &= Q_3 - \frac{a^2 b^{-2} \dot{q}_3(-\dot{q}_1 Q_1 - \dot{q}_2 Q_2 + a^2 b^{-2} \dot{q}_3 Q_3)}{\dot{q}_1^2 + \dot{q}_2^2 + a^4 b^{-4} \dot{q}_3^2} \end{aligned} \right\} \quad (4.10)$$

Differentiating the first integral (4.3) with respect to t , noting that Yerugin function is zero since n is an arbitrary constant, we have

$$m(\dot{q}_1 \ddot{q}_1 + \dot{q}_2 \ddot{q}_2 + \dot{q}_3 \ddot{q}_3) + mg\dot{q}_3 = 0 \quad (4.11)$$

substituting (4.10) into (4.11), and using the constraint condition (4.2), we get

$$\dot{q}_1 Q_1 + \dot{q}_2 Q_2 + \dot{q}_3 Q_3 + mg\dot{q}_3 = 0 \quad (4.12)$$

This is the algebraic equation corresponding to (3.6).

Finally, from (3.9), we get a solution

$$Q_1=0, \quad Q_2=0, \quad Q_3=-mg \quad (4.13)$$

The problem ends in the inverse problem of the classical Appell's example^[4].

If a special integral of this problem is given as

$$\omega = \dot{q}_1 / \dot{q}_2 = 1 \quad (4.14)$$

Then (3.6) becomes

$$Q_1 \dot{q}_2 - Q_2 \dot{q}_1 - \Phi = 0 \quad (4.15)$$

or

$$Q_1 \dot{q}_2 - Q_2 \dot{q}_1 - \Phi_1 = 0 \quad (4.16)$$

where

$$\Phi_1 = m \dot{q}_1^2 \Phi \quad (4.17)$$

And (3.8) becomes

$$-Q_2 - \frac{\partial \Phi_1}{\partial \dot{q}_1} = 0, \quad Q_1 - \frac{\partial \Phi_1}{\partial \dot{q}_2} = 0, \quad -\frac{\partial \Phi_1}{\partial \dot{q}_3} = 0 \quad (4.18)$$

Substituting the first two equations of (4.18) into (4.16), we have

$$\frac{\partial \Phi_1}{\partial \dot{q}_1} \dot{q}_1 + \frac{\partial \Phi_1}{\partial \dot{q}_2} \dot{q}_2 - \Phi_1 = 0 \quad (4.19)$$

According to the last equation in (4.18), the generalized solution of (4.19) is

$$\Phi_1 = \dot{q}_2 \varphi \left(\frac{\dot{q}_1}{\dot{q}_2}, q_1, q_2, q_3, t \right) \quad (4.20)$$

where φ is an arbitrary function. However, Yerugin function satisfy

$$\Phi|_{\dot{q}_1/\dot{q}_2=1} = 0$$

i.e.,

$$\varphi(1, q_1, q_2, q_3, t) = 0 \quad (4.21)$$

We may take, e. g.,

$$\Phi_1 = (\dot{q}_2 - \dot{q}_1) \psi(q_1, q_2, q_3, t) \quad (4.22)$$

where ψ is an arbitrary function. Then by (4.18), we get

$$Q_1 = Q_2 = \psi(q_1, q_2, q_3, t) \quad (4.23)$$

So, given a special integral (4.14), it is impossible to find all the $Q_s (s=1, 2, 3)$. Moreover, Q_1 and Q_2 are arbitrary functions satisfying (4.23).

Example 2 A sledge is sliding on a horizontal plane. The projection on the plane of the center of mass of the sledge coincides with the contact point of the sledge with the plane. Let m and J be the mass of it and the moment of inertia of it with respect to the center of mass, respectively. The position of this system is determined by three parameters: the coordinates (x, y) of the projection on the plane of the center of mass and the angle θ between the symmetric axis of the sledge and a fixed axis Ox . The kinetic energy of the system is

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J \dot{\theta}^2 \quad (4.24)$$

and the nonholonomic constraint is

$$f = \dot{y} - x \dot{\theta} g \theta = 0 \quad (4.25)$$

Given a first integral of the system

$$\omega = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + V(x, y, \theta) = h \quad (4.26)$$

Suppose that the generalized forces Q_x , Q_y , and Q_θ depend only on the coordinates. We are to determine those generalized forces.

Let $q_1 = x$, $q_2 = y$, and $q_3 = \theta$. By some computation, (3.6) becomes the following equation

$$q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + \frac{\partial V}{\partial q_1} \dot{q}_1 + \frac{\partial V}{\partial q_2} \dot{q}_2 + \frac{\partial V}{\partial q_3} \dot{q}_3 = 0 \quad (4.27)$$

Since h is an arbitrary constant, $\Phi = 0$. Now by (3.9), we have

$$Q_1 + \frac{\partial V}{\partial q_1} = 0, \quad Q_2 + \frac{\partial V}{\partial q_2} = 0, \quad Q_3 + \frac{\partial V}{\partial q_3} = 0.$$

Hence the generalized forces are

$$Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2}, \quad Q_3 = -\frac{\partial V}{\partial q_3} \quad (4.28)$$

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