

THE NEW CRITERIA OF ELASTIC AND FATIGUE FAILURE IN THE COMPONENT OF COMPLEX STRESS STATES

Hu Zhu-hua (胡铸华)

(Changsha Communications Institute, Changsha)

(Received Oct. 3, 1987; Communicated by Hsueh Dah-wei)

Abstract

In this paper, a total criterion on elastic and fatigue failure in complex stress, that is, octahedral stress strength theory on dynamic and static states on the basis of studying modern and classic strength theories. At the same time, an analysis of an independent and fairly comprehensive theoretical system is set up. It gives generalized failure factor by 36 materials and computational theory of the 11 states of complex stresses on a point, and derives the operator equation on generalized allowable strength and a computational method for a total equation can be applied to dynamic and static states. It is illustrated that the method has a good exactness through computation of eight examples of engineering. Therefore, the author suggests applying it to engineering widely.

Key words the strength theory on octahedral stresses, the generalized failure factor of materials, tension-compression ratio, yield-strength ratio, the factor of engineering design, the operator $[\sigma_r]$ for generalized allowable strength of materials

I. The Source of the New Strength Theory

Now let $\sigma_1 \geq \sigma_2 \geq \sigma_3$, be principal stress at dangerous point in the component, " σ_r " be the generalized allowable limit of materials when the stress state in the component is axis, which is divided by national stipulated factor of safety " n ". The item is named generalized allowable strength operator $[\sigma_r]$. S_H is the strength of complex stress, and $H_\nu = \nu$ is Poisson's ratio.

The maximum principal strain theory by Saint-Venant can be developed into

$$S_H = \sigma_1 - H_\nu(\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (1.1)$$

Assume that the strength limit of simple tensile (or compressive) is separately expressed with σ_+ , σ_- , then $H_\sigma = \sigma_+ / \sigma_-$, is the ratio of tension-compression (See Table 1).

Table 1 Tension-compression ratio H_σ

Material trademark	9XC	P9	P18	Y12	40X	HT15-33	HT20-40	Aluminium alloys	Grey cast iron	Cast steel
H_σ	0.42	0.48	0.48	0.41	0.51	0.23	0.2824	0.67	0.333	0.667~0.8

Coulomb-Mohr's modified shear stress theory can be developed into

$$S_H = \sigma_1 - H \sigma_3 \leq [\sigma_r] \quad (1.2)$$

Assuming σ_y is yield limit, σ_t is strength limit, then $H_y = \sigma_y / \sigma_t$ is yield-strength ratio. From Eq. (1.1), we obtain

$$S_H = \sigma_1 - H_y (\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (1.3)$$

Synthesizing Eqs. (1.1)–(1.3) we draw an equation on unitized failure theory to outline

$$S_H = \sigma_1 - H (\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (1.4)$$

$$H = H = H(0, H_y, 0.5^*, 0.683, 0.73^*, H_o, H_y, \dots, 1, \dots, 1^*)$$

$$r = r(-1, \dots, 0, 0.1, 0.15, \dots, 1)$$

In this equation H represents the coefficient of material generalized failure, which can be elected arbitrarily from mechanical properties of materials. r is equal to the ratio of minimum stress σ_{\min} to the maximum stress σ_{\max} on the component. Its yield surface can be represented in a unique manner as the octahedral stress $(\sigma_{oct}, \tau_{oct})^{(1)}$, or the stress invariant

$$I_\sigma = \frac{1}{3} \sigma_{kk}, \quad \sqrt{I_2} = \left(\frac{1}{2} S_{ij} S_{ij} \right)^{1/2}$$

$$I_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki}, \quad S_{ij} = \sigma_{ij} - \delta_{ij} I_\sigma$$

$$\omega_\sigma = \frac{1}{3} \arcsin(-3\sqrt{3} I_3 / (2I_2^{3/2}))^{(1)} \quad \left(-\frac{\pi}{6} < \omega_\sigma < \frac{\pi}{6} \right)$$

and thus we find

$$S_{oct} = A \sigma_{oct} + B \tau_{oct} \leq K_r \quad (a)$$

where $A = 1 - 2H$, $B = \sqrt{2}(1+H)\cos(\omega_\sigma)$, $K_r = \sigma_r$, or

$$S_H = A_1 I_\sigma + B_1 \sqrt{2I_2} \leq K_r \quad (b)$$

$$A_1 = 1 - 2H, \quad B_1 = 2(1+H)\cos(\omega_\sigma) / \sqrt{3}$$

If materials are subjected to the combination of direct and shear stresses, from Eqs. (1.4) or (a), (b), the design equation in the system is given by

$$S_H = \frac{1}{2} [(1-H)\sigma + (1+H)(\sigma^2 + 4\tau^2)^{1/2}] \leq [\sigma_r] \quad (1.5)$$

$$H = (-1^*, 0, \sigma=0, \tau=0, H_y, H_o, H_y, 0.5, 0.683, \dots, 1)$$

This theory states that failure can be assumed to occur when the maximum stress strength S_H in the complex stress system becomes equal to that at the allowable stress limit in the single axial to test σ_r . The criterion is referred to as the generalized stress strength S_H theory.

It is also referred to as octahedral (shear and direct) stress strength theory.

This theory considers elastic-plastic and fatigue failure. Put all those factors of elasticity, ductility, plastic-flow and brittle fracture together with the stress of the midst. It is distinctive with the classic idea. Practically speaking, they vary in material. Their external forces are various, and they are also distinguished in the absorption of energy. The plastic strain (alt) is always present in the important region subjected to loading on component, and therefore, it is

necessary to take into account. Its explanation on geometry is that six pairs of planes are cut. It becomes a parallelepipedon with an equilateral oblique angle. The isogonic coordinates axes and symmetric into stress states of space.

II. Analysis of Total Criterion

A. For brittle material

The lateral strain is small, or plastic deflection is very small when $H=0$, like sudden fracture (materials such as glasses, ceramics, rocks, plain concrete, and cast iron). From Eqs. (1.4), we obtain Rankine's maximum principal stress theory and develop it into

$$S_0 = \sigma_1 \leq [\sigma_r] \quad (2.1)$$

When $H = H_r = \nu$, from Eqs. (1.4), we obtain E. Mariotto's maximum principal strain theory and once again develop it into

$$S_r = \sigma_1 - H_r(\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (2.2)$$

When $H_c = \sigma_+ / \sigma_-$, from Eqs. (1.4), we obtain tension-compression ratio theory:

$$S_c = \sigma_1 - H_c(\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (2.3)$$

When $\sigma_2 = 0$, from Eqs. (2.3), we obtain Coulomb-Mohr's theory and once again develop it into

$$S_2 = \sigma_1 - H_c \sigma_3 \leq [\sigma_r] \quad (2.4)$$

When $H_c = 1$, for plastic material, from Eq. (2.4), we obtain Tresca theory and once again develop it into

$$S_{1,3} = \sigma_1 - \sigma_3 \leq [\sigma_r] \quad (2.5)$$

For pure shear, we have tensile, yielding limit, $\sigma_1 = -\sigma_3 = -\tau$, and from Eq. (2.4) we obtain

$$S_\tau = (1 + H_c)\tau \leq [\sigma_r] \quad (2.6)$$

For torsion, we have to take account of concentrated factors of stress K_τ , from Eq. (2.6), and we find strength discriminant for brittle materials

$$S_b = K_\tau(1 + H_c)\tau \leq [\sigma_r] \quad (2.7)$$

For $\sigma_{\max} > 0$, $\sigma_{\min} < 0$, when the component is subjected to simultaneous application of σ and τ , we obtain the equation on tensile-compression principal stress of brittle materials (see Eqs. (1.5))

$$S_c = \frac{1}{2} [(1 - H_c)K_\sigma \sigma + (1 + H_c)\sqrt{(K_\sigma \sigma)^2 + (2K_\tau \tau)^2}] \leq [\sigma_r] \quad (2.8)$$

$$S_c = \frac{1}{2} [(1 - H_c)K_\sigma \sigma - (1 + H_c)\sqrt{(K_\sigma \sigma)^2 + (2K_\tau \tau)^2}] \leq [\sigma_r] \quad (2.9)$$

When $\tau = 0$, we obtain stress strength condition for tension-compression or bending condition only

$$S_{bs} = K_\sigma \sigma_{\max} \leq [\sigma_r] \quad (2.10)$$

B. For plastic materials

Let H_p stand for the standard of difficulty-easiness on plastic-flow of materials. H values can be determined from many experiments. The results are presented in Table 2.

Table 2 Yield-intensity factors H_p

Material trademark	A2	A3	A5	35	45	Bridge steel 16Q	A3Q
H_p	0.559~0.524	0.6316~0.60	0.56~0.53	0.593	0.59	0.605	0.6315
Material trademark	14MnNb	15MnTi	15MV	16Mn	50B	20CrMn	2Cr13
H_p	0.682	0.625~0.75	0.75~0.68	0.673	0.6875	0.789	0.6818
Material trademark	1Cr17	Brone Be	HP659-1	Cast iron ZG40Mn	Cast iron QD60-2	QD45-5	LY-12
H_p	0.625	0.685	0.692	0.666	0.70	0.735	0.665
Material trademark	Carbon steel 15A	ZGCr-17	20Mn2	Steel mild	Cast iron	30CrMn SiNiA	
H_p	0.603	0.625	0.75	0.6	0.67	0.844	

From Eqs. (1.4), we obtain generalized plastic strength theory^[5]:

$$S = \sigma_1 - H_p(\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (2.11)$$

On the other hand, by experiment, we want to use the optimum method to find $H_p = 0.683$. The factor that applies to ductile materials (e.g. Table 2) the substitutes H_p value for Eq. (2.11), and obtains generalized plastic strength criteria

$$S_p = \sigma_1 - 0.683(\sigma_2 + \sigma_3) \leq [\sigma_r] \quad (2.12)$$

or expresses stress invariants I_1 and I_2

$$S_p = A_1 \sqrt{I_2} - B_1 I_1 \leq K_r \quad (2.13)$$

where

$$A_1 = 1.943 \cos(\omega_\sigma)^{1/2}, \quad B_1 = 0.368, \quad K_r = \sigma_r$$

Now let us further analyse complex stress states on a point. "+" stands for direct stress and "-" stands for compressive stress. Then

Table 3 The states of complex stress on a point

States	1	2	3	4	5	6	7	8	9	10	11
σ_1	+	-	+	+	+	+	+	+	+	+	0
σ_2	+	-	+	+	0	0	-	-	+	0	-
				$\sigma_2 > \sigma_1/2$				$ \sigma_2 > \sigma_1/2$			
σ_3	+	-	-	0	0	-	-	-	0	-	-
							$ \sigma_3 \geq 3\sigma_1 $	$ \sigma_3 > \sigma_1/2$			

States (1.1)-(2.1) belong to shear-tension type of principal stress states and are primal with tension, and the brittle fracture is increased. Taken as ideal brittle, then middle principal stress has no influence on yield states^[1]. Take $H=0$, and $H_c=1$.

From Eqs. (1.4) we find

$$S_0 = \sigma_1 \leq [\sigma_r]$$

From Eq.(2.5), we find

$$S_{1,3} = \sigma_1 - \sigma_3 \leq [\sigma_r]$$

States (2.2)–(2.6) belong to shear-compression, and the flowability is increased. Take $H_s = 0.683$ (or see Table 2). Computing the strength of ocmplex stress for (2.2)–(2.6), from Eqs. (1.4) we may use Eq. (2.12)¹⁵.

$$S_p = \sigma_1 - 0.683(\sigma_2 + \sigma_3) \leq [\sigma_r]$$

Considering the component subjected to combined stress on σ and τ , by Eq. (2.2) for structure steel we find

$$S' = 0.35\sigma + 0.65\sqrt{\sigma^2 + 4\tau^2} \leq [\sigma_r] \quad (2.14)$$

From Eqs. (1.4) or (1.5) we can obtain design equation

$$S_{\sigma\tau} = \frac{1}{2}[(1-H)\sigma + (1+H)\sqrt{\sigma^2 + (2\tau)^2}] \leq [\sigma_r] \quad (2.15)$$

where

$$H = H(-1^*, 0, \sigma=0, \tau=0, H_s, H_c, H_p, 0.5, 0.683, 1, \dots)$$

We can take any value H in the bracket by different materials and stress states. Now analyse Eqs. (2.15).

When $H=0$, we obtain the developing theory of maximum principal stress theory

$$S'_{\sigma\tau} = \frac{1}{2}[\sigma + \sqrt{\sigma^2 + 4\tau^2}] \leq [\sigma_r] \quad (2.16)$$

and when $\sigma=0$, or $\tau=0$, we obtain maximum stress strength theory from Eq. (2.16)

$$\begin{aligned} \tau_{\max} &\leq [\tau_r], \\ \sigma_{\max} &\leq [\sigma_r] \end{aligned} \quad (2.17)$$

When $H=1$, for the stress states of wholly plastic or ductile materials, we obtain equivalence equation

$$\begin{aligned} S'_{\sigma\tau} &= \sqrt{\sigma^2 + (2\tau)^2} \leq [\sigma_r] \\ S'_{\sigma\tau} &\Longleftrightarrow S_{Tresca} = \sigma_1 - \sigma_3 \leq [\sigma_r] \end{aligned} \quad (2.18)$$

When $H=0.5$, taking the body as incompressible, whose volume is constant, we obtain the developing formula of twi-shear stress

$$S' = \frac{1}{4}[\sigma + 3\sqrt{\sigma^2 + (2\tau)^2}] \leq [\sigma_r] \quad (2.19)$$

When $H=H$, we obtain equivalence equation with Von-Mises theoretical values

$$\begin{aligned} S'_{\sigma\tau} &= \frac{1}{2}[(1-H_p)\sigma + (1+H_p)\sqrt{\sigma^2 + (2\tau)^2}] \leq [\sigma_r] \\ S_{\sigma\tau} &\Longleftrightarrow S_{\text{Von, Mises}} = (\sigma^2 - 3\tau^2)^{1/2} \leq [\sigma_r]^{1/2} \end{aligned} \quad (2.20)$$

When $H=0.683$, then

$$S = 0.1586\sigma + 0.8415\sqrt{\sigma^2 + (2\tau)^2} \leq [\sigma_r] \quad (2.21)$$

The above Eqs. (2.14)–(2.21) are applied to computation of static and fatigue failure of dynamics.

Whether in static or dynamic states, for the strength computation of ductile materials, the author suggests that we should make use of the generalized plastic criteria

$$S_H = \sigma_1 - 0.683(\sigma_2 + \sigma_3) \leq [\sigma_r]$$

and

$$S_{\sigma\tau} = 0.16\sigma + 0.84 \sqrt{\sigma^2 + (2\tau)^2} \leq [\sigma_r] \quad (2.21)$$

Thanks to their difficulty-easiness, more time can be saved than other theories, and they are exact enough.

III. Intensity Operator σ_r and Design Coefficient N_H

There is a σ_r term in the above 26 equations. That σ_r is derived according to Goodman-Gerber-Sooderberg Curved Line^[3,4]. Take account of engineering factor K'_σ (or K''_σ). The σ_r is known as intensity operator. It can be applied to any circle on generalized stress (static or dynamic). $\exists \forall S \in (\sigma, \tau)$, then

$$S = \frac{1}{2}[1 + (1 - r_s)\sin(\omega t + \phi) + r_s]S_{\max} \quad (3.1)$$

Then

$$\sigma_r = K'_\sigma \sigma \quad (3.2)$$

where

$$\begin{aligned} \sigma &= \sigma(\sigma_s, \sigma_I, \sigma_e, \dots) \\ K'_\sigma &= \frac{2C_{k\sigma}}{1 + (1 + r_\sigma)C_{k\sigma}\psi_\sigma - r_\sigma} \\ C_{k\sigma} &= \frac{\varepsilon_\sigma \beta_k \beta}{\alpha_\sigma K_\sigma} \\ r_\sigma &= \pm \frac{\sigma_{\min}}{\sigma_{\max}} = (-1, \dots, 0, 0.1, 0.2, \dots, 1). \end{aligned} \quad (3.3)$$

where K'_σ is the engineering total factor of design, $C_{k\sigma}$ is the component factor, σ is an important material property limit for members subjected to static loading or dynamic loading, ε_σ is the dimension factor of component, β_k is the enviromental factor of the component in the working, β is the quality factor, α_σ is a factor sometimes necessary to be considered (e.g. oil hole, without it $\alpha_\sigma = 1$), K_σ is the stress concentrated factor, ψ_σ is a sensitive factor of materials, σ_I is ultimate strength, σ_e is the endurance limit, and r_σ is the ratio of the minimum stress to the maximum stress.

τ of the pure shear or torsion also has many terms, that is,

$$\begin{aligned} \tau_r &= K'_\tau \tau \\ \tau &= \tau(\tau_s, \tau_I, \tau_e, \dots) \\ K'_\tau &= \frac{2C_{k\tau}}{1 + (1 + r_\tau)C_{k\tau}\psi_\tau - r_\tau} \end{aligned} \quad (3.4)$$

$$r_\tau = \pm \frac{\tau_{\min}}{\tau_{\max}} = r_\tau (-1, \dots, 0, 0.1, \dots, 1)$$

When $K_\tau \gg K_\sigma$, we may use $K_{\tau\sigma} = (K_\tau + K_\sigma)/2$ as K_σ to find $N_{\sigma\tau} = \sigma_\tau / S_H$. All the above factors may be found from mechanical engineering handbooks (e.g.[2]).

Let N_H be design coefficient in the component working, which is equal to the ratio of the generalized stress limit σ_r to the generalized stress strength under complex stress states, but greater than or equal to the factor of safety by the national stipulation "n".

$$N_H = \sigma_r / S_H \quad \sigma_r = \sigma_r(\sigma_y, \sigma_t, \sigma_e, \dots, \sigma_\tau) \quad (3.5)$$

$$N_H \geq n \quad S_H = S_H(S_\sigma, S_\sigma, S_k, S_p, \dots) \quad (3.6)$$

where σ_r , S_H and r may be selected by the needs of engineering design, and the components will be subjected to external forces, and so will the materials of components.

IV. Application

(1) The complex stress states (for plastic materials)

c.g.	σ_1	σ_2	σ_3	Mises theory S_2	Tresca theory S_3	Two-shear stress S_t	This theory Eq. (2.12) S_p	$\frac{S_2 - S_4}{S_4} = \Delta_{4t}$	$\frac{S_2 - S_1}{S_4} = \Delta_{4t}$	$\left \frac{S_4 - S_p}{S_4} \right = \Delta_{4p}$
1	100	-300	-1000	964.4	1100	750	987.9	0.141	0.222	0.024
2	-500	-700	-1600	1615	1100	650	1070	0.081	0.359	0.05
3	600	0	-500	954	1100	850	952	0.153	0.109	0.002

(2) The component subjected to combined stresses for bending and torsion

c.g.	σ	τ	Material trademark	H_p	Mises theory S_2	This theory Eq. (2.21)' S_H	$\Delta_{44} = \frac{S_H - S_2}{S_4}$
4	164	35.4	16Mn	0.673	175	176.24 ↑	0.007
5	195	61.3	30CrMnSiNiA	0.84	222	224.7 ↑	0.012

(3) Computation on fatigue strength of shaft on dynamics

c.g.	σ	τ	σ_e	τ_e	r_σ	r_τ	K_σ	ε_σ	β_s	β	α_σ	α_τ	K_τ	ε_τ	ψ_τ	σ_r
6	± 50	$\begin{matrix} +40 \\ +20 \end{matrix}$	220	120	-1	+0.5	1.93	0.88	1.0	1.0	1.0	1.0	1.46	0.81	0.1	100.3
7	± 122	$\begin{matrix} +81.5 \\ 0 \end{matrix}$	540	310	-1	0	1.65	0.73	1.0	1.0	1.0	1.0	1.35	0.78	0.1	238.9
8	± 35.5	$\begin{matrix} +28 \\ 0 \end{matrix}$	410	240	-1	0	2.13	0.88	1.0	0.77	1.0	1.0	2.06	0.81	0.05	130.43
c.g.	Eq. (2.21)' $S_H = 0.16\sigma + 0.84\sqrt{\sigma^2 + 4\tau^2}$				$N_H = \frac{\sigma_r}{S_H}$			$n_\sigma^{[2]}$	$n_\tau^{[2]}$	$n_{\sigma\tau}^{[2]}$	$\Delta = \left \frac{N_H - n_{\sigma\tau}}{n_{\sigma\tau}} \right $					
6	53.139				1.88			2.00	5.71	1.88	0					
7	142.764				1.674			1.96	4.16	1.77	0.05					
8	59.11				2.232			2.44	5.82	2.22	0.008					

V. Conclusion

The above generalized unitized theory on dynamic and static states can be applied to brittle materials, ductile materials, brittle fracture, and plastic flow states, as well as complex

stress states for dynamic and static states, and it can also be used to derive the classic strength theory and two-shear stress theory, and developed for dynamic states. When this theory is employed for engineering, it has good exactness. Its computation is easier than Von. Mises' and it is better than Cepechko theory. Therefore, the author of this article suggests using it in engineering.

For different section, make allowances for the equation with Z_σ , Z_τ , e.g. Eqs. (1.5)

$$S_H = \frac{1}{2} [(1-H)Z_\sigma\sigma + (1+H)\sqrt{(Z_\sigma\sigma)^2 + (2Z_\tau\tau)^2}] \leq [\sigma_r] \quad (5.1)$$

round beam $Z_\sigma=1.0$, $Z_\tau=1.0$, solid round shaft $Z_\tau=0.7216$, thin wall pipe $Z_\sigma=0.769$ ($d_1/d_2=0.95$), $Z_\tau=1.00$, an axle "I" beam ($h_1/h=0.90$), $Z_\sigma=0.8475$.

References

- [1] Kachanov, L. M., *Plastic Theoretical Basis*, translated by Zhou Cheng-ti and Du Qing-hua, People Educational Publishers (1983). (Chinese version)
- [2] Cepechko C. B., et al., *Material Strength Handbook*, Moscow (1957). (Chinese version)
- [3] Kocanda, S., *Fatigue Failure of Metals*, Sijthoff & Noord-hoff International Publishers (1978).
- [4] Hu Zhu-hua, A Research to Develop R.Von. Mises Distortion Energy Theory and the Theory of Dynamic Spacial Shear Stress Strength of Materials, *Applied Mechanics*, **2** (1989), 637.
- [5] Hu Zhu-hua, Generalized strength theory in plastics, Scientific Report Lecture in Beijing University of Aero. and Astro. (1985). (in Chinese)