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A UNIFORM HIGH-ORDER METHOD FOR A SINGULAR PERTURBATION PROBLEM IN CONSERVATIVE FORM

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Abstract

A uniform high-order method is presented for the numerical solution of a singular perturbation problem in conservative form. We firest replace the original second-order problem (1.1) by two equivalent first-order problems (1.4), i.e., the solution of (1.1) is a linear combination of the solutions of (1.4). Then we derive a uniformly $O(h^{m+1})$ accurate scheme for the first-order problems (1.4), where m is an arbitrary nonnegative integer, so we can get a uniformly $O(h^{m+1})$ accurate solution of the original problem (1.1) by relation (1.3). Some illustrative numerical results are also given.

Key words uniform high-order method, singular perturbation problem, initial value problem

I. Introduction

A number of finite difference schemes or finite element methods were proposed for the numerical solution of singular perturbation problems in recent decades. Especially, uniform arbitrary order difference schemes were constructed for the selfadjoint and the nonselfadjoint singularly perturbed ordinary differential equations in [3,7 – 10]. We may observe, however, that the difference scheme is very complicate if its convergence order is larger than two.

In this paper we will derive a simple uniform high-order method for the following nonselfadjoint singular perturbation problem in conservation form.

$$\left.\begin{aligned}&\varepsilon u''+(a(x)u)'=f(x), \quad \text{for } 0<x<1,\\&u(0)=\mu_0,\quad u(1)=\mu_1,\end{aligned}\right\}\tag{1.1}$$

where ε is a parameter in (0,1]; μ_0 and μ_1 are given boundary data; the coefficient $a(x)$ is in $W^{m+1}\equiv\{F: F(x)\in C^m[0,1],\ F^{(m)}\in \mathrm{Lip}1\}$ and $f(x)$ in W^m where m is an arbitrary nonnegative integer, and $a(x)>\alpha>0$ for some positive constant α.

The basic idea of deriving our uniform high-order method may be described as follows. Let $u_i(x), i=1,2$ be the solutions of the following second-order initial value problems:

$$\left.\begin{aligned}&\varepsilon u_i''(x)+(a(x)u_i(x))'=F_i(x), \quad \text{for } 0<x<1,\\&u_i(0)=D_i,\quad u_i'(0)=E_i,\end{aligned}\right\}\tag{1.2}$$

where $F_1(x)=f(x)$, $F_2(x)\equiv 0$, $E_1=D_2=0$, $D_1=\mu_0$ and $E_2=1/\varepsilon$. Obviously the solution of (1.1) can be represented as

$$u(x)=u_1(x)+\gamma u_2(x),$$

where $\gamma=(\mu_1-u_1(1))/u_2(1)$. Note that

$$u_2(1)=\varepsilon^{-1}\int_0^1 \exp\left[-\int_t^1 \varepsilon^{-1}a(s)ds\right]dt$$
$$\geqslant A(1-\exp[-A/\varepsilon])\geqslant M>0 ,$$

where the positive constant A is defined by $A=\max\limits_{0<x<1} a(x)$ and M may be chosen as $M=A(1-\exp[-A])$. Therefore, if we have got approximate solutions $u_i^h(x)$, $i=1, 2$ such that

$$|u_i^h(x)-u_i(x)|\leqslant Mh^p, \qquad p>0 ,$$

then

$$u^h(x)\equiv u_1^h(x)+(\mu_1-u_1^h(1))/u_2^h(1)\cdot u_2^h(x) \tag{1.3}$$

satisfies

$$|u^h(x)-u(x)|\leqslant Mh^p ,$$

where (and throughout this paper) $M, M_1, \ldots$ will be used to denote generic constants independent of ε and the discretization mesh h. Noting that (1.2) may be converted into the following first-order initial value problems

$$\left.\begin{aligned}&\varepsilon u_i'(x)+a(x)u_i(x)=G_i(x) \qquad (i=1,2)\\ &u_i(0)=D_i ,\end{aligned}\right\} \tag{1.4}$$

where

$$G_i(x)=\int_0^x F_i(x)dx+\varepsilon E_i+a(0)D_i ,$$

we can get the solution of the original problem (1.1) through solving two first-order initial value problems (1.4).

In sections II and III, we will construct a family of arbitrary order convergence schemes, uniformly in the parameter ε , for the initial value problems (1.4), so we also get an arbitrary order accurate solution for the original problem (1.1) by relation (1.3). It's evident that our method is simpler than those in [3,7 – 10] due to the integration of equation (1.2). Some numerical results will be given in section IV.

II. An Exact Scheme

In this section we will consider the following general form of problems (1.4):

$$\left.\begin{aligned}&Lu\equiv\varepsilon u'+a(x)u=\tilde{h}(x), \quad \text{for } 0<x<1\\ &u(0)=\mu ,\end{aligned}\right\} \tag{2.1}$$

and derive an exact finite difference scheme for (2.1).

Let N be a positive integer and define the uniform mesh length $Nh=1$. Let the grid points $\{x_i\}$be given by $x_i=ih,\ i=0,1,\cdots,N$, and denote by u_i^h the approximate value (to be determined) for $u_i=u(x_i)$.

Because the solution of (2.1) is uniquely determined by the right-hand side and an initial value condition, we may represent the solution of (2.1) as

$$u(x)=v_1^i(x)u(x_i)+v_0^i(x), \quad \text{for } x_i\leqslant x\leqslant x_{i+1}, \tag{2.2}$$

where $v_k^i(x), k=0,1$ satisfy

$$\left.\begin{aligned} &Lv_k^i(x)=\tilde{h}_k(x), \qquad k=0,1, \text{ for } x_i\leqslant x\leqslant x_{i+1},\\ &v_0^i(x_i)=0,\ v_1^i(x_i)=1, \end{aligned}\right\} \tag{2.3}$$

where $\tilde{h}_1(x)\equiv 0$ and $\tilde{h}_0(x)=\tilde{h}(x)$. Let $x=x_i+sh,\ w_k^i(s)=v_k^i(x_i+sh)\ (k=0,1)$, and $\tau=h/\varepsilon$. Then (2.2) becomes

$$u(x_i+sh)\equiv w_1^i(s)u(x_i)+w_0^i(s), \qquad s\in[0,1], \tag{2.4}$$

where $w_k^i(s),\ k=0,1$ are the solutions of

$$\left.\begin{aligned} &lw_k^i\equiv dw_k^i/ds+\tau a(x_i+sh)w_k^i=\tau\tilde{h}_k(x_i+sh),\\ &\qquad\qquad s\in(0,1],\ k=0,1,\\ &w_0^i(0)=0,\ w_1^i(0)=1. \end{aligned}\right\} \tag{2.5}$$

From the maximum principle it follows that for $0\leqslant s\leqslant 1$,

$$0< w_1^i(s)\leqslant \exp[-\alpha\tau s], \tag{2.6}$$

and

$$|w_0^i(s)|\leqslant M(1-\exp[-\alpha\tau s]), \tag{2.7}$$

where $M=\alpha^{-1}\max\limits_{0\leqslant x\leqslant 1}|\tilde{h}(x)|$.

If we set $s=1$ in (2.4),then we get the following exact scheme for (2.1):

$$\left.\begin{aligned} &u_{i+1}=w_1^i(1)u_i+w_0^i(1), \qquad i=0,1,\cdots,N-1,\\ &u_0=\mu. \end{aligned}\right\} \tag{2.8}$$

This scheme has property

$$|u_i|\leqslant|\mu|+\sigma^{-1}\max_{0\leqslant j\leqslant N-1}|w_0^j(1)|, \qquad i=0,1,\cdots,N, \tag{2.9}$$

where $\sigma=1-\exp[-\alpha\tau]$. In fact, from (2.6) – (2.8),we have

$$\begin{aligned} |u_{i+1}|&=\left|\sum_{j=0}^{i}w_0^j(1)\prod_{l=j+1}^{i}w_1^l(1)+\mu\prod_{j=0}^{i}w_1^j(1)\right|\\ &\leqslant\max_{0\leqslant j\leqslant i}|w_0^j(1)|\sum_{j=0}^{i}\exp[-\alpha\tau j]+|\mu|\\ &\leqslant|\mu|+\sigma^{-1}\max_{0\leqslant j\leqslant N-1}|w_0^j(1)|. \end{aligned}$$

Since there are not, in general, explicit expressions for functions $w_k^i(s)$, the exact scheme

(2.8) is not applicable. Thus, we will propose a truncated difference scheme which may be as accurate as we hope. We first consider the following scheme

$$\left.\begin{aligned} &Z_{i+1}=A_1^iZ_i+A_0^i, \qquad i=0,1,\cdots,N-1, \\ &Z_0=\mu, \end{aligned}\right\} \tag{2.10}$$

where A_k^i, $k=0,1$ are approximations of $w_k^i(1)$ such that

$$|A_1^i-w_1^i(1)|\leqslant\nu_1\leqslant\sigma/2, \quad |A_0^i-w_0^i(1)|\leqslant\nu_0. \tag{2.11}$$

About scheme (2.10) we have

Lemma 1 Let u and $\{Z_i\}$ be the solutions of (2.1) and (2.10). Then

$$|Z_i|\leqslant|\mu|+(\sigma-\nu_1)^{-1}\max_{0<i<N-1}|A_0^i|, \tag{2.12}$$

$$|Z_i-u(x_i)|\leqslant\sigma^{-1}(\nu_0+\nu_1(|\mu|+(\sigma-\nu_1)^{-1}\max_{0<i<N-1}|A_0^i|)). \tag{2.13}$$

Proof Analogous to the proof of (2.9) we have

$$|Z_{i+1}|\leqslant|\mu|+(1-A)^{-1}\max_{0<i<N-1}|A_0^i|,$$

where $A=\max\limits_i|A_0^i|$. From the assumption (2.11) it's easy to get $1-A\geqslant\sigma-\nu_1$. Thus (2.12) follows.

To prove (2.13), let $y_i=Z_i-u_i$. From (2.8) and (2.10) we have $y_0=0$, and for $i=0$, $1\ldots, n-1$,

$$\begin{aligned} y_{i+1}-w_1^i(1)y_i&=Z_{i+1}-w_1^i(1)Z_i-w_0^i(1) \\ &=Z_{i+1}-w_1^i(1)Z_i-(Z_{i+1}-A_1^iZ_i)+A_0^i-w_0^i(1) \\ &=(A_1^i-w_1^i(1))Z_i+(A_0^i-w_0^i(1)). \end{aligned}$$

Therefore (2.13) follows from (2.9) and (2.12). The Lemma is proved.

III. A Truncated Scheme

We may see from Lemma 1 that the approximate difference scheme (2.10) is high-order accurate if the coefficients A_k^i, $k=0,1$ are high-order approximations of $w_k^i(1)$ in (2.8). In order to approximate $w_k^i(1)$, let

$$w_k^i(s)=\sum_{n=0}^{m}h^nw_{kn}^i(s)+h^{m+1}r_{km}^i(s), \qquad k=0,1. \tag{3.1}$$

By Tayor expansion of $a(x_i+sh)$ and $\tilde{h}_k(x_i+sh)$ about $s=0$, we get from equations (2.5) the following recursive relations for $w_{kn}^i(s)$: (for simplicity, we omit superscript i)

$$\left.\begin{aligned} &lw_{kn}\equiv dw_{kn}/ds+a(x_i)\tau w_{kn}=\tau H_{kn}, \qquad s\in(0,1], \\ &\qquad k=0,1, \ n=0,1,\cdots,m, \\ &w_{00}(0)=0, \ w_{10}(0)=1, \ w_{kn}(0)=0, \ \text{for } n\geqslant1; \ k=0,1, \end{aligned}\right\} \tag{3.2}$$

where $H_{k0}=\tilde{h}_{k0}$, and for $n>1$,

$$H_{kn}=\tilde{h}_{kn}s^n-\sum_{j=0}^{n-1}a_{n-j}s^{n-j}w_{kj}, \tag{3.3}$$

coefficients $\tilde{h}_{kn}$ and a_n are defined by the formula $y_n=y^{(n)}(x_i)/n!$. The remainder term r_{km} satisfies

$$lr_{km}=dr_{km}/ds+a(x_i+sh)\tau r_{km}=\tau\phi_{km}, \qquad s\in(0,1], \qquad (3.4)$$
$$r_{km}(0)=0,$$

where

$$\phi_{km}=h^{-1}(\bar{\tilde{h}}_{km}-\tilde{h}_{km})s^m-\sum_{n=0}^{m}h^{-1}(\bar{a}_{m-n}-a_{m-n})s^{m-n}w_{kn}, \qquad (3.5)$$

$\bar{a}_n=a_n(\theta_i)$, $x_i<\theta_i<x_{i+1}$, and $\bar{\tilde{h}}_{km}$ is similarly defined.

The following lemma is a basic result for our future analysis.

Lemma 2 If $y(s)$ is in $C'[0,1]$ such that $y(0)=0$ and $|ly|\leqslant M_1\tau$ (or $|\tilde{l}y|\leqslant M_1\tau$), then

$$|y(s)|\leqslant M\sigma(s)\leqslant M\sigma, \qquad (3.6)$$

where $\sigma(s)=1-\exp[-a\tau s]$

Proof Because

$$l\sigma(s)=a\tau\exp[-a\tau s]+a(x_i+sh)\tau(1-\exp[-a\tau s])\geqslant a\tau,$$

and $\tilde{l}\sigma(s)\geqslant a\tau$, we may choose $a^{-1}M_1(1-\exp[-a\tau s])$ as a barrier function for $y(s)$ and get, from the maximum principle,

$$|y(s)|\leqslant a^{-1}M_1(1-\exp[-a\tau s]),$$

which completes the proof of Lemma 2.

Now we analyze the functions $w_{kn}(s)$ and $r_{km}(s)$ in (3.1).

Lemma 3 If $a(x)$, $\tilde{h}_k(x)$ are in w^{m+1}, then

$$|w_{kn}(s)|\leqslant M\sigma(s)\leqslant M\sigma, \qquad n=1,2,\cdots,m;\ k=0,1, \qquad (3.7)$$
$$|r_{km}(s)|\leqslant M\sigma(s)\leqslant M\sigma, \qquad k=0,1, \qquad (3.8)$$

Proof From the maximum principle we have $|w_{10}(s)|\leqslant 1$, $|w_{00}(s)|\leqslant M\sigma$. Thus $|H_{k1}|\leqslant M_1$, $k=0,1$, and one may obtain (3.7) for $n=1$ by Lemma 2. Assume inductively (3.7) holds for $1\leqslant j\leqslant n-1$. Then we have $|H_{kn}|\leqslant M_2$ which follows (3.7) for $n=j$ by Lemma 2. Hence (3.7) is proved by inductive method. Noting that $a(x)$, $\tilde{h}_k(x)$ is in W^{m+1} and thus $|\phi_{km}|\leqslant M$, we get (3,8) by Lemma 2 again.

Lemma 3 is proved.

It's evident that equation (3.2) have explicit solutions. Hence we discard the remainder terms $h^{m+1}r_{km}$ and get the following truncated scheme

$$\left.\begin{aligned}&u^k_{i+1}=A^m_{1i}u^k_i+A^m_{0i}, \qquad i=0,1,\cdots,N-1,\\&u^k_0=\mu,\end{aligned}\right\} \qquad (3.9)$$

where

$$A^m_{ki}=\sum_{n=0}^{m}h^n w^i_{kn}(1), \qquad k=0,1.$$

Theorem 1 Assume $a(x)>\alpha>0$ and $\tilde{h}(x)$ are in W^{m+1}. Let $u(x)$ and $\{u_i^h\}$ be the solutions of (2.1) and (3.9). Then

$$\max_{0<i<N} |u_i^h - u(x_i)| \leqslant Mh^{m+1}.$$

Proof From Lemma 3 we have

$$|A_{ki}^m - w_k^i(1)| = h^{m+1}|r_{km}^i(1)| \leqslant \nu_k \equiv M_1 h^{m+1}\sigma, \qquad k=0,1,$$

where h is assumed to be sufficiently small such that $M_1 h^{m+1}\sigma \leqslant \sigma/2$. Noting that

$$|A_{0i}^m| \doteq \left|\sum_{n=0}^{m} h^n w_{0n}^i(1)\right| \leqslant \sum_{n=0}^{m} h^n M\sigma,$$

we apply Lemma 1 to obtain

$$\begin{aligned} |u_i^h - u(x_i)| &\leqslant \sigma^{-1}\left(\nu_0 + \nu_1(\sigma-\nu_1)^{-1} \max_{0<i<N-1} |A_{0i}^m|\right) \\ &\leqslant \sigma^{-1}(\nu_0 + M_2\nu_1) \leqslant Mh^{m+1}, \end{aligned}$$

which completes the proof of Theorem 1.

Now we have constructed a uniform high-order scheme for the numerical solution of the first-order initial value problem (1.4). Therefore, we can get a uniform high-order accurate approximation for the solution of the original problem (1.1) by the technique introduced in section I. The details are described in the following theorem.

Theorem 2 Assume $a(x)>\alpha>0$ is in W^{m+1} and $f(x)$ in W^m. Let u be the solution of (1.1) and $\{u_{1j}^h\}$, $\{u_{2j}^h\}$ the solutions of scheme (3.9) for problems (1,4). Then we have

$$\max_{0<j<N} |u_j^h - u(x_j)| \leqslant Mh^{m+1},$$

where $u_j^h = u_{1j}^h + (\mu_1 - u_{1N}^h)/u_{2N}^h \cdot u_{2j}^h$.

IV. Numerical Results

In this section we present some numerical results to illustrate Theorem 2 that have been previously discussed. The problem on which the numerical experiments were conducted was

$$\left.\begin{aligned} &\varepsilon u'' + u' = f(x,\varepsilon), \qquad 0<x<1, \\ &u(0)=\mu_0,\ u(1)=\mu_1, \end{aligned}\right\} \tag{4.1}$$

where the right-hand side $f(x,\varepsilon)$ and the boundary data μ_0, μ_1 were determined by the exact solution $u(x)$ of equation (4.1)

$$u(x) = \sin x + (\exp[-x/\varepsilon] - \exp[-1/\varepsilon])/(1 - \exp[-1/\varepsilon]).$$

In order to conveniently generate a wide variation of ε and h, problem (4.1) was run with $\varepsilon \equiv h^s$ for various values of s . For each value of s the mesh length h was successively halved starting with $h=1/8$ and ending with $h=1/256$. The numerical results are presented in Tables 1 – 3 where we only list the maximum error $E_\infty \equiv \max_{0<i<N} |u_i^h - u(x_i)|$ and the numerical rate of convergence $\mathrm{Rate} \equiv (\ln E_\infty^1 - \ln E_\infty^2)/\ln 2$, where E_∞^1 and E_∞^2 are the maximum errors

corresponding to $h=1/N$ and $h=1/(2N)$.

We can see from the tables that the numerical rates of convergence agree fairly well with the analytically predicted ones.

Table 1 Numerical results for uniform high-order method with $m=0$

	$\varepsilon=h^{1/2}$		$\varepsilon=h$		$\varepsilon=h^{3/2}$		$\varepsilon=h^2$		$\varepsilon=h^{5/2}$	
N	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate
8	1.6E-2		2.6E-2		3.6E-2		4.5E-2		4.9E-2	
		0.77		0.79		0.77		0.83		0.87
16	9.5E-3		1.5E-2		2.1E-2		2.5E-2		2.7E-2	
		0.81		0.91		0.85		0.91		0.94
32	5.4E-3		8.0E-3		1.2E-2		1.4E-2		1.4E-2	
		0.86		0.96		0.90		0.96		0.97
64	3.0E-3		4.1E-3		6.2E-3		7.0E-3		7.1E-3	
		0.90		0.98		0.93		0.98		0.99
128	1.6E-3		2.1E-3		3.3E-3		3.5E-3		3.6E-3	
		0.93		0.99		0.96		0.99		0.99
256	8.3E-4		1.0E-3		1.7E-3		1.8E-3		1.8E-3	

Table 2 Numerical results for uniform high-order method with $m=1$

	$\varepsilon=h^{1/2}$		$\varepsilon=h$		$\varepsilon=h^{3/2}$		$\varepsilon=h^2$		$\varepsilon=h^{5/2}$	
N	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate
8	6.4E-4		1.6E-3		3.0E-3		4.6E-3		5.4E-3	
		1.59		1.67		1.65		1.73		1.83
16	2.1E-4		5.1E-4		9.6E-4		1.4E-3		1.5E-3	
		1.66		1.81		1.77		1.87		1.93
32	6.8E-5		1.5E-4		2.8E-4		3.8E-4		4.0E-4	
		1.74		1.89		1.84		1.94		1.97
64	2.0E-5		3.9E-5		7.9E-5		9.8E-5		1.0E-4	
		1.81		1.94		1.89		1.97		1.99
128	5.8E-6		1.0E-5		2.1E-5		2.5E-5		2.5E-5	
		1.84		1.97		1.92		1.98		1.99
256	1.6E-6		2.6E-6		5.6E-6		6.3E-6		6.4E-6	

Table 3 Numerical results for uniform high-order method with $m=2$

	$\varepsilon=h^{1/2}$		$\varepsilon=h$		$\varepsilon=h^{3/2}$		$\varepsilon=h^2$		$\varepsilon=h^{5/2}$	
N	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate
8	2.2E-5		3.7E-5		5.9E-5		9.0E-5		1.1E-4	
		2.79		2.77		2.65		2.65		2.75
16	3.1E-6		5.4E-6		9.4E-6		1.4E-5		1.6E-5	
		2.84		2.90		2.74		2.81		2.89
32	4.4E-7		7.3E-7		1.4E-6		2.0E-6		2.2E-6	
		2.90		2.96		2.80		2.90		2.95
64	5.9E-8		9.4E-8		2.0E-7		2.7E-7		2.8E-7	
		3.11		2.98		2.85		2.95		2.98
128	6.8E-9		1.2E-8		2.8E-8		3.5E-8		3.6E-8	
				2.93		2.89		2.97		2.99
256	—		1.5E-9		3.8E-9		4.5E-9		4.5E-9	

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