

## NONLINEAR STRAIN COMPONENTS OF GENERAL SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS

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### Abstract

*On the basis of nonlinear strain component formulations of three-dimensional continuum, this paper has derived the nonlinear strain component formulations of shells with initial geometric imperfections. The derivation is not confined to a special shell, therefore they possess general properties. These formulations provide the theoretical basis of the strain analysis for geometric nonlinear problems of shells with initial geometric imperfections*

**Key words** initial geometric imperfections, shells, nonlinear, strain

### I. Introduction

Generally there are initial geometric imperfections in varying degrees due to the deviation of construction in the practical shells used in engineering, though they are designed according to some ideal curved surfaces. Some serious imperfections can be the main reason of making products scrapped and important structures collapse<sup>[1]</sup>.

Up to now, linear problems of shells with initial geometric imperfections have been studied in great detail, such as Chen's small parameter method<sup>[2]</sup>, Lu and Gao's perturbation method<sup>[3]</sup>, Han, Gould<sup>[4]</sup> and Kato Shiro's finite element method<sup>[5]</sup>, and so on. But nonlinear analysis of shells has not been discussed very much<sup>[6, 7]</sup>, and nonlinear problems with initial geometric imperfections have been studied much less<sup>[8]</sup>. The nonlinear strain component formulations of imperfect shells presented in this paper can be the theoretical basis of the strain analysis for geometric nonlinear problems of shells with initial geometric imperfections. In contrast to the approaches used generally, which are based on the middle surface deformation theory, this article has derived nonlinear geometric equations of shells with initial geometric imperfections, based on nonlinear geometric equations of three-dimensional continuum and considering initial geometric imperfections as initial displacements. As a proof, these general strain formulations are degenerated into the strain components of plates and shallow shells with initial displacements<sup>[8]</sup>, and also into linear and nonlinear strain components of ideal shells<sup>[6, 9, 10, 12]</sup>. The initial imperfections of shells in this paper can be finite.

### II. Nonlinear Geometric Equations of Three-Dimensional Continuum

Consider initial geometric imperfection as initial displacements. Both initial displacements and

deformed displacements are assumed to be finite. According to total Lagrangian formulations<sup>[11]</sup>, strain tensor in time  $t + \Delta t$  can be written as:

$$\begin{aligned} {}^{t+\Delta t}\varepsilon_{ij}^* &= \frac{1}{2}({}_0^tU_{i,j} + {}_0^tU_{j,i} + {}_0^tU_{k,j}{}_0^tU_{k,i} + {}_0^tU_{i,j} + {}_0^tU_{j,i} + {}_0^tU_{k,i}{}_0^tU_{k,j}) \\ &+ \frac{1}{2}({}_0^tU_{k,j}{}_0^tU_{k,i} + {}_0^tU_{k,i}{}_0^tU_{k,j}) \end{aligned} \quad (2.1)$$

Let  $t=0$  be initial time, and  ${}_0^tU_i, {}_0^tU_i (i=x,y,z)$  be the initial and deformed displacement components respectively. Using eqs. (2.1) we can obtain:

$$\begin{aligned} \Delta^t\varepsilon_{ij}^* &= \frac{1}{2}({}_0^tU_{i,j} + {}_0^tU_{j,i} + {}_0^tU_{k,j}{}_0^tU_{k,i} + {}_0^tU_{i,j} + {}_0^tU_{j,i} + {}_0^tU_{k,i}{}_0^tU_{k,j}) \\ &+ \frac{1}{2}({}_0^tU_{k,j}{}_0^tU_{k,i} + {}_0^tU_{k,i}{}_0^tU_{k,j}) \end{aligned}$$

Initial displacements do not result in any initial strains, for the initial displacements concerned are initial deviation to ideal shells, hence:

$$\frac{1}{2}({}_0^tU_{i,j} + {}_0^tU_{j,i} + {}_0^tU_{k,j}{}_0^tU_{k,i}) = 0$$

Then strain tensor  $\{\Delta^t\varepsilon_{ij}^*\}$  consist of following three parts:

$$\{\Delta^t\varepsilon_{ij}^*\} = L(V) + NL(V) + IL(V_0, V)$$

where  $L$ ,  $NL$ , and  $IL$  represent linear, nonlinear and coupling parts respectively.

Replacing  $\Delta^t\varepsilon_{ij}^*, {}_0^tU_x, {}_0^tU_y, {}_0^tU_z$  and  ${}_0^tU_x, {}_0^tU_y, {}_0^tU_z$  with  $\varepsilon_{ij}, U_0, V_0, W_0$  and  $U, V, W$ , respectively, introducing the following notations:

$$\left. \begin{aligned} e_{xx} &= \frac{\partial U}{\partial x}, \quad e_{yy} = \frac{\partial V}{\partial y}, \quad e_{zz} = \frac{\partial W}{\partial z} \\ e_{xy} &= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}, \quad e_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}, \quad e_{yz} = \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \\ 2\omega_x &= \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}, \quad 2\omega_y = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}, \quad 2\omega_z = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \end{aligned} \right\} \quad (2.2)$$

then the strain components with initial geometric imperfections can be written as:

$$\begin{aligned} \varepsilon_{xx} &= e_{xx} + \frac{1}{2} \left[ e_{xx}^2 + \left( \frac{1}{2}e_{xy} + \omega_x \right)^2 + \left( \frac{1}{2}e_{xz} - \omega_y \right)^2 \right] + e_{xx}e_{0xx} \\ &+ \left( \frac{1}{2}e_{0xy} + \omega_{0x} \right) \left( \frac{1}{2}e_{xy} + \omega_x \right) + \left( \frac{1}{2}e_{0xz} - \omega_{0y} \right) \left( \frac{1}{2}e_{xz} - \omega_y \right) \\ \varepsilon_{yy} &= e_{yy} + \frac{1}{2} \left[ e_{yy}^2 + \left( \frac{1}{2}e_{xy} - \omega_x \right)^2 + \left( \frac{1}{2}e_{yz} + \omega_z \right)^2 \right] + e_{yy}e_{0yy} \\ &+ \left( \frac{1}{2}e_{0xy} - \omega_{0x} \right) \left( \frac{1}{2}e_{xy} - \omega_x \right) + \left( \frac{1}{2}e_{0yz} + \omega_{0z} \right) \left( \frac{1}{2}e_{yz} + \omega_z \right) \end{aligned}$$

$$\begin{aligned}
e_{zz} &= e_{zz} + \frac{1}{2} \left[ e_{zz}^2 + \left( \frac{1}{2} e_{zz} + \omega_y \right)^2 + \left( \frac{1}{2} e_{yz} - \omega_z \right)^2 \right] + e_{zz} e_{0zz} \\
&\quad + \left( \frac{1}{2} e_{0zz} - \omega_y \right) \left( \frac{1}{2} e_{zz} + \omega_y \right) + \left( \frac{1}{2} e_{0yz} - \omega_z \right) \left( \frac{1}{2} e_{yz} - \omega_z \right) \\
\gamma_{zy} &= e_{zy} + e_{zz} \left( \frac{1}{2} e_{zy} - \omega_z \right) + e_{yy} \left( \frac{1}{2} e_{zy} + \omega_z \right) + \left( \frac{1}{2} e_{zz} - \omega_y \right) \left( \frac{1}{2} e_{yz} + \omega_z \right) \\
&\quad + e_{0zz} \left( \frac{1}{2} e_{zy} - \omega_z \right) + e_{yy} \left( \frac{1}{2} e_{0zy} + \omega_{0z} \right) + \left( \frac{1}{2} e_{0zz} - \omega_y \right) \left( \frac{1}{2} e_{yz} + \omega_z \right) \\
&\quad + e_{zz} \left( \frac{1}{2} e_{0zy} - \omega_{0z} \right) + e_{0yy} \left( \frac{1}{2} e_{zy} + \omega_z \right) + \left( \frac{1}{2} e_{zz} - \omega_y \right) \left( \frac{1}{2} e_{0yz} + \omega_{0z} \right) \\
\gamma_{yz} &= e_{yz} + e_{yy} \left( \frac{1}{2} e_{yz} - \omega_z \right) + e_{zz} \left( \frac{1}{2} e_{yz} + \omega_z \right) + \left( \frac{1}{2} e_{zy} - \omega_z \right) \left( \frac{1}{2} e_{zz} + \omega_y \right) \\
&\quad + e_{0yy} \left( \frac{1}{2} e_{yz} - \omega_z \right) + e_{zz} \left( \frac{1}{2} e_{0yz} + \omega_{0z} \right) + \left( \frac{1}{2} e_{0zy} - \omega_z \right) \left( \frac{1}{2} e_{zz} + \omega_y \right) \\
&\quad + e_{yy} \left( \frac{1}{2} e_{0yz} - \omega_{0z} \right) + e_{0zz} \left( \frac{1}{2} e_{yz} + \omega_z \right) + \left( \frac{1}{2} e_{zy} - \omega_z \right) \left( \frac{1}{2} e_{0zz} + \omega_{0y} \right) \\
\gamma_{zx} &= e_{zx} + e_{zz} \left( \frac{1}{2} e_{zx} + \omega_y \right) + e_{zz} \left( \frac{1}{2} e_{zx} - \omega_y \right) + \left( \frac{1}{2} e_{zy} + \omega_z \right) \left( \frac{1}{2} e_{yz} - \omega_z \right) \\
&\quad + e_{0zz} \left( \frac{1}{2} e_{zx} + \omega_y \right) + e_{zz} \left( \frac{1}{2} e_{0zz} - \omega_{0y} \right) + \left( \frac{1}{2} e_{0zy} + \omega_{0z} \right) \left( \frac{1}{2} e_{yz} - \omega_z \right) \\
&\quad + e_{zz} \left( \frac{1}{2} e_{0zz} + \omega_{0y} \right) + e_{0zz} \left( \frac{1}{2} e_{zx} - \omega_y \right) + \left( \frac{1}{2} e_{zy} + \omega_z \right) \left( \frac{1}{2} e_{0yz} - \omega_{0z} \right)
\end{aligned} \tag{2.3}$$

in which concrete expressions of  $e_{0zz}, \dots, 2\omega_{0z}$  can be obtained from eqs. (2.2) where the deformed displacements  $U, V$  and  $W$  must be replaced with initial displacements  $U_0, V_0$  and  $W_0$ .

We can obtain expression of  $e_{zz}, \dots, 2\omega_z$  in orthogonal curvilinear coordinate system<sup>[12]</sup>:

$$\begin{aligned}
e_{11} &= \frac{1}{H_1} \frac{\partial U_\alpha}{\partial \alpha} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \beta} U_\beta + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \gamma} U_\gamma, \\
e_{22} &= \frac{1}{H_2} \frac{\partial U_\beta}{\partial \beta} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \gamma} U_\gamma + \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial \alpha} U_\alpha, \\
e_{33} &= \frac{1}{H_3} \frac{\partial U_\gamma}{\partial \gamma} + \frac{1}{H_1 H_3} \frac{\partial H_3}{\partial \alpha} U_\alpha + \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial \beta} U_\beta, \\
e_{13} &= \frac{H_1}{H_3} \frac{\partial}{\partial \gamma} \left( \frac{U_\alpha}{H_1} \right) + \frac{H_3}{H_1} \frac{\partial}{\partial \alpha} \left( \frac{U_\gamma}{H_3} \right) \\
e_{23} &= \frac{H_3}{H_2} \frac{\partial}{\partial \beta} \left( \frac{U_\gamma}{H_3} \right) + \frac{H_2}{H_3} \frac{\partial}{\partial \gamma} \left( \frac{U_\beta}{H_2} \right) \\
e_{12} &= \frac{H_2}{H_1} \frac{\partial}{\partial \alpha} \left( \frac{U_\beta}{H_2} \right) + \frac{H_1}{H_2} \frac{\partial}{\partial \beta} \left( \frac{U_\alpha}{H_1} \right) \\
2\omega_1 &= \frac{1}{H_2 H_3} \left[ \frac{\partial}{\partial \beta} (H_3 U_\gamma) - \frac{\partial}{\partial \gamma} (H_2 U_\beta) \right]
\end{aligned} \tag{2.4}$$

$$\left. \begin{aligned} 2\omega_2 &= \frac{1}{H_1 H_3} \left[ \frac{\partial}{\partial \gamma} (H_1 U_\alpha) - \frac{\partial}{\partial \alpha} (H_3 U_\gamma) \right] \\ 2\omega_3 &= \frac{1}{H_1 H_2} \left[ \frac{\partial}{\partial \alpha} (H_2 U_\beta) - \frac{\partial}{\partial \beta} (H_1 U_\alpha) \right] \end{aligned} \right\}$$

where  $H_1$ ,  $H_2$  and  $H_3$  are Lamé's coefficients,  $U_\alpha$ ,  $U_\beta$  and  $U_\gamma$  are the deformed displacement projection on the three coordinate axes in curvilinear coordinate system  $(\alpha, \beta, \gamma)$  at any point.

Expressions of  $e_{011}$ , ...,  $2\omega_{03}$ , can be obtained by substituting  $U_\alpha$ ,  $U_\beta$ ,  $U_\gamma$  with initial displacements  $U_{0\alpha}$ ,  $U_{0\beta}$ ,  $U_{0\gamma}$  respectively in eqs. (2.4). So we can write the general expressions of nonlinear strain components of three-dimensional continuum in orthogonal curvilinear coordinate system easily:

$$\left. \begin{aligned} \varepsilon_{11} &= e_{11} + \frac{1}{2} \left[ e_{11}^2 + \left( \frac{1}{2} e_{12} + \omega_3 \right)^2 + \left( \frac{1}{2} e_{13} - \omega_2 \right)^2 \right] + e_{11} e_{011} \\ &\quad + \left( \frac{1}{2} e_{012} + \omega_{03} \right) \left( \frac{1}{2} e_{12} + \omega_3 \right) + \left( \frac{1}{2} e_{013} - \omega_{02} \right) \left( \frac{1}{2} e_{13} - \omega_2 \right) \\ \varepsilon_{22} &= e_{22} + \frac{1}{2} \left[ e_{22}^2 + \left( \frac{1}{2} e_{12} - \omega_3 \right)^2 + \left( \frac{1}{2} e_{23} + \omega_1 \right)^2 \right] + e_{22} e_{022} \\ &\quad + \left( \frac{1}{2} e_{012} - \omega_{03} \right) \left( \frac{1}{2} e_{12} - \omega_3 \right) + \left( \frac{1}{2} e_{023} + \omega_{01} \right) \left( \frac{1}{2} e_{23} + \omega_1 \right) \\ \varepsilon_{33} &= e_{33} + \frac{1}{2} \left[ e_{33}^2 + \left( \frac{1}{2} e_{13} + \omega_2 \right)^2 + \left( \frac{1}{2} e_{23} - \omega_1 \right)^2 \right] + e_{33} e_{033} \\ &\quad + \left( \frac{1}{2} e_{013} + \omega_{02} \right) \left( \frac{1}{2} e_{13} + \omega_2 \right) + \left( \frac{1}{2} e_{023} - \omega_{01} \right) \left( \frac{1}{2} e_{23} - \omega_1 \right) \\ \gamma_{12} &= e_{12} + e_{11} \left( \frac{1}{2} e_{12} - \omega_3 \right) + e_{22} \left( \frac{1}{2} e_{12} + \omega_3 \right) + \left( \frac{1}{2} e_{13} - \omega_2 \right) \left( \frac{1}{2} e_{23} + \omega_1 \right) \\ &\quad + e_{011} \left( \frac{1}{2} e_{12} - \omega_3 \right) + e_{22} \left( \frac{1}{2} e_{012} + \omega_{03} \right) + \left( \frac{1}{2} e_{013} + \omega_{02} \right) \left( \frac{1}{2} e_{23} + \omega_1 \right) \\ &\quad + e_{11} \left( \frac{1}{2} e_{012} - \omega_{03} \right) + e_{022} \left( \frac{1}{2} e_{12} + \omega_3 \right) + \left( \frac{1}{2} e_{13} - \omega_2 \right) \left( \frac{1}{2} e_{023} + \omega_{01} \right) \\ \gamma_{23} &= e_{23} + e_{22} \left( \frac{1}{2} e_{23} - \omega_1 \right) + e_{33} \left( \frac{1}{2} e_{23} + \omega_1 \right) + \left( \frac{1}{2} e_{12} - \omega_3 \right) \left( \frac{1}{2} e_{13} + \omega_2 \right) \\ &\quad + e_{022} \left( \frac{1}{2} e_{23} - \omega_1 \right) + e_{33} \left( \frac{1}{2} e_{023} + \omega_{01} \right) + \left( \frac{1}{2} e_{012} - \omega_{03} \right) \left( \frac{1}{2} e_{13} + \omega_2 \right) \\ &\quad + e_{22} \left( \frac{1}{2} e_{023} - \omega_{01} \right) + e_{033} \left( \frac{1}{2} e_{23} + \omega_1 \right) + \left( \frac{1}{2} e_{12} - \omega_3 \right) \left( \frac{1}{2} e_{013} + \omega_{02} \right) \\ \gamma_{31} &= e_{13} + e_{11} \left( \frac{1}{2} e_{13} + \omega_2 \right) + e_{33} \left( \frac{1}{2} e_{13} - \omega_2 \right) + \left( \frac{1}{2} e_{12} + \omega_3 \right) \left( \frac{1}{2} e_{23} - \omega_1 \right) \\ &\quad + e_{011} \left( \frac{1}{2} e_{13} + \omega_2 \right) + e_{33} \left( \frac{1}{2} e_{013} - \omega_{02} \right) + \left( \frac{1}{2} e_{012} + \omega_{03} \right) \left( \frac{1}{2} e_{23} - \omega_1 \right) \\ &\quad + e_{11} \left( \frac{1}{2} e_{013} + \omega_{02} \right) + e_{033} \left( \frac{1}{2} e_{13} - \omega_2 \right) + \left( \frac{1}{2} e_{12} + \omega_3 \right) \left( \frac{1}{2} e_{023} - \omega_{01} \right) \end{aligned} \right\} \quad (2.5)$$

### III. Nonlinear Strain Components of General Shells with Initial Geometric Imperfections

Let  $\alpha_1$  and  $\alpha_2$  be the Gauss curvilinear coordinates;  $\xi$  be the line coordinate which is perpendicular to the middle surface of shell.  $\alpha_1$ ,  $\alpha_2$  and  $\xi$  are orthogonal with each other. Lamé's coefficients of curve surface are:

$$H_1 = A_1 \left( 1 + \frac{\xi}{R_1} \right), \quad H_2 = A_2 \left( 1 + \frac{\xi}{R_2} \right), \quad H_3 = 1 \quad (3.1)$$

Substituting eqs. (3.1) into eqs. (2.4), we obtain:

$$\left. \begin{aligned} e_{11} &= \frac{1}{1+\xi/R_1} \left( \frac{1}{A_1} \frac{\partial U^\xi}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} V^\xi + \frac{W^\xi}{R_1} \right) \\ e_{22} &= \frac{1}{1+\xi/R_2} \left( \frac{1}{A_2} \frac{\partial V^\xi}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} U^\xi + \frac{W^\xi}{R_2} \right), \quad e_{33} = \frac{\partial W^\xi}{\partial \xi} \\ e_{12} &= \frac{1}{1+\xi/R_1} \left( \frac{1}{A_1} \frac{\partial V^\xi}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} U^\xi \right) \\ &\quad + \frac{1}{1+\xi/R_2} \left( \frac{1}{A_2} \frac{\partial U^\xi}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} V^\xi \right) \\ e_{13} &= \frac{\partial U^\xi}{\partial \xi} + \frac{1}{1+\xi/R_1} \left( \frac{1}{A_1} \frac{\partial W^\xi}{\partial \alpha_1} - \frac{U^\xi}{R_1} \right) \\ e_{23} &= \frac{\partial V^\xi}{\partial \xi} + \frac{1}{1+\xi/R_2} \left( \frac{1}{A_2} \frac{\partial W^\xi}{\partial \alpha_2} - \frac{V^\xi}{R_2} \right) \\ 2\omega_1 &= -\frac{\partial V^\xi}{\partial \xi} + \frac{1}{1+\xi/R_2} \left( \frac{1}{A_2} \frac{\partial W^\xi}{\partial \alpha_2} - \frac{V^\xi}{R_2} \right) \\ 2\omega_2 &= \frac{\partial U^\xi}{\partial \xi} - \frac{1}{1+\xi/R_1} \left( \frac{1}{A_1} \frac{\partial W^\xi}{\partial \alpha_1} - \frac{U^\xi}{R_1} \right) \\ 2\omega_3 &= \frac{1}{1+\xi/R_1} \left( \frac{1}{A_1} \frac{\partial V^\xi}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} U^\xi \right) \\ &\quad - \frac{1}{1+\xi/R_2} \left( \frac{1}{A_2} \frac{\partial U^\xi}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} V^\xi \right) \end{aligned} \right\} \quad (3.2)$$

in which  $U^\xi$ ,  $V^\xi$  and  $W^\xi$  are the deformed displacements in the direction of coordinate lines  $\alpha_1$ ,  $\alpha_2$  and  $\xi$  for any point of the shell. The expressions of  $e_{011}, \dots, 2\omega_{03}$  can be got by replacing the initial displacements of the corresponding shell with deformed displacements in eqs. (3.2).

So using eqs. (2.5), the strain components of any point in the shell can be expressed in curvilinear coordinate system easily.

Let deformed displacements of the shell be:

$$U^\xi = U + \xi v, \quad V^\xi = V + \xi \psi, \quad W^\xi = W + \xi \chi \quad (3.3)$$

where  $U$ ,  $V$  and  $W$  are the deformed displacements of the middle surface of the shell.

Using basic assumptions of thin shells, i. e.  $\gamma_{13}=0$ ,  $\gamma_{23}=0$ ,  $e_{33}=0$ , and considering  $1+\xi/R_1 \approx 1$ ,  $1+\xi/R_2 \approx 1$ , let

$$\left. \begin{aligned} *e_{11} &= \frac{1}{A_1} \frac{\partial U}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} V + \frac{W}{R_1} \\ *e_{22} &= \frac{1}{A_2} \frac{\partial V}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} U + \frac{W}{R_2} \\ *e_{12} &= \frac{1}{A_1} \frac{\partial V}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} U, \quad *e_{13} = \frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} - \frac{U}{R_1} \\ *e_{21} &= \frac{1}{A_2} \frac{\partial U}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} V, \quad *e_{23} = \frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} - \frac{V}{R_2} \end{aligned} \right\} \quad (3.4)$$

Replacing the displacements in eqs. (3.4) with corresponding initial displacements we can obtain the expressions of  $*e_{011}, \dots, *e_{023}$  respectively, so we obtain

$$\left. \begin{aligned} v(1+*e_{11}+*e_{011})+\psi(*e_{12}+*e_{012})+\chi(*e_{13}+*e_{013})+*e_{13} &= 0 \\ \psi(1+*e_{22}+*e_{022})+v(*e_{21}+*e_{021})+\chi(*e_{23}+*e_{023})+*e_{23} &= 0 \\ v^2+\psi^2+(1+\chi)^2 &= 1 \end{aligned} \right\} \quad (3.5)$$

Under the condition of small strain, we have:

$$\left. \begin{aligned} v &= -(*e_{13}+*e_{013})(1+*e_{22}+*e_{022})+(*e_{23}+*e_{023})(*e_{12}+*e_{012}) \\ \psi &= -(*e_{23}+*e_{023})(1+*e_{11}+*e_{011})+(*e_{13}+*e_{013})(*e_{21}+*e_{021}) \\ \chi &= (*e_{11}+*e_{011})+(*e_{22}+*e_{022})+(*e_{11}+*e_{011})(*e_{22}+*e_{022}) \\ &\quad -(*e_{12}+*e_{012})(*e_{21}+*e_{021}) \end{aligned} \right\} \quad (3.6)$$

Let

$$\left. \begin{aligned} \kappa_{11} &= \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \psi + \frac{\chi}{R_1} \\ \kappa_{22} &= \frac{1}{A_2} \frac{\partial \psi}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v + \frac{\chi}{R_2} \\ \kappa_{12} &= \frac{1}{A_1} \frac{\partial \psi}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v, \quad \kappa_{13} = \frac{1}{A_1} \frac{\partial \chi}{\partial \alpha_1} - \frac{v}{R_1} \\ \kappa_{21} &= \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \psi, \quad \kappa_{23} = \frac{1}{A_2} \frac{\partial \chi}{\partial \alpha_2} - \frac{\psi}{R_2} \end{aligned} \right\} \quad (3.7)$$

Substitute eqs. (3.2) into the expressions of  $\varepsilon_{11}$ ,  $\varepsilon_{22}$  and  $\gamma_{12}$  in eqs. (2.5). The terms with the second power parts of  $\xi$  are neglected, considering eqs. (3.3), (3.4), (3.7), then the strain expressions of shells with initial geometric imperfections can be written as:

$$\begin{aligned} \varepsilon_{11} &= *e_{11} + \frac{1}{2}(*e_{11}^2 + *e_{12}^2 + *e_{13}^2) + *e_{11}*e_{011} + *e_{12}*e_{012} + *e_{13}*e_{013} \\ &\quad + \xi[(1+*e_{11})\kappa_{11} + *e_{12}\kappa_{12} + *e_{13}\kappa_{13} + *e_{011}\kappa_{11} + *e_{012}\kappa_{12} + *e_{013}\kappa_{13}] \\ \varepsilon_{22} &= *e_{22} + \frac{1}{2}(*e_{21}^2 + *e_{22}^2 + *e_{23}^2) + *e_{21}*e_{021} + *e_{22}*e_{022} + *e_{23}*e_{023} \\ &\quad + \xi[(1+*e_{22})\kappa_{22} + *e_{21}\kappa_{21} + *e_{23}\kappa_{23} + *e_{022}\kappa_{22} + *e_{021}\kappa_{21} + *e_{023}\kappa_{23}] \\ \gamma_{12} &= *e_{12} + *e_{21} + *e_{11}*e_{21} + *e_{22}*e_{12} + *e_{13}*e_{23} + *e_{011}*e_{21} + *e_{012}*e_{22} \\ &\quad + *e_{013}*e_{23} + *e_{021}*e_{11} + *e_{022}*e_{12} + *e_{023}*e_{13} \end{aligned} \quad (3.8)$$

$$+ \xi [(1 + *e_{22})K_{12} + (1 + *e_{11})K_{21} + *e_{21}K_{11} + *e_{12}K_{22} + *e_{23}K_{13} + *e_{13}K_{23}] \\ + \xi [*e_{021}K_{11} + *e_{022}K_{12} + *e_{023}K_{13} + *e_{011}K_{21} + *e_{012}K_{22} + *e_{013}K_{23}]$$

Therefore strain components at any point of the shell can be expressed by the strains of the middle surface of the shell.

As a proof, these general strain formulations are degenerated into the strain components of plates and shallow shells with initial displacements<sup>[8]</sup>, and also into linear and nonlinear strain components of ideal shells<sup>[6, 9, 10, 12]</sup>.

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