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VARIATIONAL PRINCIPLES FOR HYDRODYNAMIC IMPACT PROBLEMS

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Abstract

We first establish the rigorous field equations of the two continuous stages before and after entering water. Then correspondently, we obtain the specific variational principles, bounded theorems, and boundary integral equations of the second stage problems. The existence of solutions are proved and the scheme of solving the solutions are provided. Finally, as a numerical example, the ship's wave resistence problem is used to demonstrate the specific application of the second stage problems and its accuracy. Then we provide a rigorous and sound theoretical basis of variational finite element method and boundary element method for calculating the accurately fundamental equations.

Key words variational principle, boundary integral equation, hydrodynamic impact problem

I. Introduction

The signification of hydrodynamic impact problems (e.g., missile entering water, ship slamming) is well-known, which they have not been solved. In ref. [1], representative for the problem before entering water, the compressible air layer and the water region are simplified as one and two dimensional respectively. Then a difference solution is obtained, in which the condition after entering water has not been considered. In ref. [2], representative for the problem after entering water, the effect of compressible air layer before entering water has not been included and moreover, only an approximately analysis solution is obtained.

These problems, containing variational interface, copulating reaction of rigid-air-water, which are separated into two stages, before and after entering water, are unsteady and nonlinear problems, and obviously very difficult. Furthermore, the signification of these problems is that their mathematic modelling involve some other problems. For example, by modifying the problem after entering water, the ship's wave resistance problem which has not been solved yet can be modelled.

The object of this research is to establish a rigorous theoretical system containing fundamental equations and theoretical basis of variational finite element method and boundary element method.

Fundamental equations are separated into two stages. The interval of $0\leqslant t\leqslant T_1$ is called the first stages from the rigid surface beginning to compress the air layer to the surface touching the water, the fundamental equation of which is improved from ref. [1] by extending from one dimension to two in air layer region and from two dimensions to three in water region. We don't consider the influence of the thickness of air layer because it is thin, so the air layer is almost a two-dimensional problem. The interval of $T_1\leqslant t\leqslant T_2$ is called the second stage from the end of the first stage to the beginning stage after entering water, the fundamental equation of which is employed from ref. [2] but includes the actual condition of the end of the first stage. In other words, the dimension of flow field is extended to the reality and the two stages of distinctive physics are copulated. Obviously, the actuality can be modelled by the above model accurately.

Furthermore the specific variational principles, bounded theorems, and the boundary integral equations for the second stage problem are established, and the existence of the solution is proved. As for a treatment, the method establishing the complementary variational principle in ref. [3] is employed, the improvement of which compared with the representaive ref. [4] is at least that we can determine the same property in stationary point for a pair of complementary functionals by requiring only one determinative formula. Finally, employing three numerical examples of ship's wave resistence[5], we demonstrate the effective application and the accuracy of the degenerate condition of the second stage problem.

The contents of this paper provide a rigorous theoretical basis of original numerical method for calculating the accurate fundamental equations.

II. Fundamental Equations

The quantities used below are all nondimensional. The (x, y) plane of Cartesian coordinates is set on the quiescent water surface and the Oz axis is directed above. A rigid body, whose shape of surface touching the water is given, impacts downward to water vertically with prescribed velocity u. Velocity potential, density, and pressure, of water are ϕ, ρ_0 and p, respectively. Velocity, velocity potential, density, pressure, and gas state constant of air layer are Λ, ϕ_1, ρ_a, p_a and k respectively. The variational water surface represented by Σ is separated into two sections, $\Sigma=\Sigma_1 \cup \Sigma_2$ (see figs. 1 -- 3)[1, 2]. In the first stage, Σ_1 represents the section touching the compressed air layer; Σ_2 represents the section touching the atmosphere. In the second stage, Σ_1 represents the section embedded into water. The intersection of Σ_1 and Σ_2 or the boundary of surface Σ_1 is $\partial\Sigma_1$. In the first stage, surfaces Σ_1 and Σ_2 are smooth and continuous on the intersection $\partial\Sigma_1$, the outward unit normal of which is $\boldsymbol{n}$. The difference between variational water surface and quiescent water surface in the first stage is ζ, the upwardness of which is positive. In the second stage ζ represents the difference between surface Σ_2 and quiescent water surface. The thickness of air layer is h. The water domain is V, the boundary of which is S, containing infinitesimal boundary S_∞, and the outward unit normal of S is $\boldsymbol{N}$.

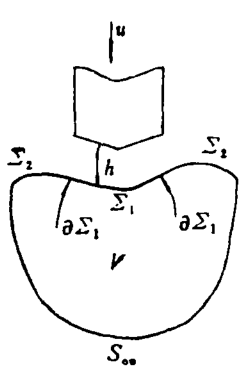

Fig. 1 The middle process of the first stage

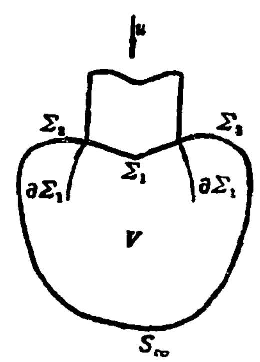

Fig. 2 The end of the first stage or the beginning of the second stage

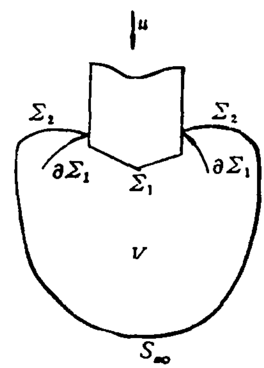

Fig. 3 The middle process of the second stage

We introduce the interface variable family ψ; $\psi=0$ is defined on the water surface Σ in the first stage and the Σ_2 in the second stage, and the domain V is defined as $\psi\geqslant 0$.

The dimensionless fundamental equations of the first stage are

$$V:\quad \Delta\phi=0 \tag{2.1}$$

$$\Sigma_1:\quad \partial(\rho_a h)/\partial t+\nabla_1\cdot(\rho_a h\Lambda)=0 \tag{2.2}$$

$$p=p_a \tag{2.3}$$

$$\Sigma_2:\quad p=p_{a0} \tag{2.4}$$

$$\Sigma:\quad \partial\psi/\partial t+\nabla\phi\cdot\nabla\psi=0 \tag{2.5}$$

$$S_\infty:\quad \nabla\phi=0 \tag{2.6}$$

$$t=0:\quad h=h_0 \tag{2.7}$$

$$t=T_1:\quad h=0 \tag{2.8}$$

$$\zeta=\zeta_{T_1} \tag{2.9}$$

$$\partial\Sigma_1:\quad \rho_a h\Lambda=\rho_{a0}\overline{(h\Lambda)} \tag{2.10}$$

and

$$V:\quad -\rho_0^{-1}p=\partial\phi/\partial t+|\nabla\phi|^2/2+gz \tag{2.11}$$

$$\Sigma_1:\quad \Lambda=\nabla_1\phi_1 \tag{2.12}$$

$$p_a=\rho_a^k \tag{2.13}$$

$$-\rho_a^{-1}p_a=(k-1)(\partial\phi_1/\partial t+\Lambda^2/2)-1 \tag{2.14}$$

$$h+\int_0^t u\,dt-h_0+\zeta=0 \tag{2.15}$$

$$t=0:\quad \phi=0 \tag{2.16}$$

The following eqs. (2.17)–(2.19) can be derived from the above. From the definition of interface family ψ we know

$$\psi=\zeta-z \tag{2.17}$$

and obtain

$$\delta\psi=\delta\zeta \tag{2.18}$$

and by substituting eq. (2.18) into (2.15), we obtain

$$\Sigma_1:\quad \delta h=-\delta\psi \tag{2.19}$$

$$(\partial V=S=\Sigma\cup S_\infty,\quad \Sigma=\Sigma_1\cup\Sigma_2)$$

where $\nabla_1=\boldsymbol{i}\partial/\partial x+\boldsymbol{j}\partial/\partial y$, ∇ is a three-dimensional operator. h_0 is the prescribed value of the air layer at the instant of behaving compressiby (i.e., the initial constant density and pressure, ρ_{a0} and p_{a0}, as well as the quiescent water surface begin varying), and it is the value of z coordinate of the section of rigid body compressing the air layer. T_1, ζ_{T_1} and $\overline{(h\Lambda)}$ are prescribed values determined below. The value on Σ_1 is determined from eqs. (2.8) and (2.15) as

$$\zeta_{T_1}=h_0-\int_0^{T_1} u\,dt \tag{2.20}$$

in which the time T_1 is determined by the maximum value of h_0 on $\partial\Sigma_1$, i.e., $h_0|_{max}$, corresponding to zero of ζ_{T_1} (see Fig. 2): $0=h_0|_{max}-\int_0^{T_1}udt$. The remainders of prescribed values, ζ_{T_1} on Σ_2 and $\overline{(h\Lambda)}$ on $\partial\Sigma_1$, are determined below.

Fundamental equations are constructed by the following reasons and the combination of the stationary condition of a functional (see below). The liquid flow equations are usual (or see [2]). The air layer equations are obtained as follows: eq. (2.2) is extended from one dimension in ref. [1] to two dimensions, Eqs. (2.12) – (2.14) are usual (see [7]), and eq. (2.15) presented here is the integration of transverse continuity condition in ref. [1] from the velocity form to the displacemnt form and is extended to two dimensions. The interface variable family ψ is introduced and the relative eqs. (2.15), (2.17) – (2.19) are presented here, by means of the variational method of mixed continua[3, 5, 6, 8] and they play important part in establishing the variational principles of rigid-air-liquidcopulated problem and can be seen later.

The remainders of the prescribed values in fundamental equations and variational functionals mentioned above are the ζ_{T_1} on Σ_2 and the $\overline{(h\Lambda)}$ on $\partial\Sigma_1$, which are not known in advance and should be determined iteratively in the following ways. From eqs. (2.4), (2.11) and (2.17) we can determine the following on Σ_2

$$\zeta_{T_1}=-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)\,|_{\Sigma_2(T_1)}-g^{-1}\rho_0^{-1}p_{a0}$$

$$\approx-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)\,|_{z=0,T=T_1}-g^{-1}\rho_0^{-1}p_{a0} \qquad (2.21)$$

and the property employed above is valid as follows.

Property 1 When the defined domain Σ of the independent variable of the height ζ of water-free surface is replaced by $z=0$, the error of ζ is infinitesimal and is allowable.

So we replace the defined domain Σ by $z=0$ in some proper place later on.

The new value of ζ_{T_1} can be determined by the value of ϕ of the last iterative procedure.

The new value of $\overline{(h\Lambda)}$ can be determined from eqs. (2.12). (2.15) and (2.17) by the value of ϕ_1, ψ of the last iterative procedure.

The convergence of the two iterative precedures is given in property 2 later on.

The fundamental equations of the second stage are

$$V:\quad \nabla\phi=0 \qquad (2.22)$$

$$\Sigma_2:\quad \partial\psi/\partial t+\nabla\phi\cdot\nabla\psi=0 \qquad (2.23)$$

$$-\rho_0^{-1}p\equiv\partial\phi/\partial t+|\nabla\phi|^2/2+gz=-\rho_0^{-1}p_{a_0} \qquad (2.24)$$

$$\Sigma_1:\quad (\nabla\phi-\boldsymbol{u})\cdot\boldsymbol{N}=0 \qquad (2.25)$$

$$S_\infty:\quad \nabla\phi=0 \qquad (2.26)$$

$$t=T_1:\quad \phi=\phi_{T_1} \qquad (2.27)$$

$$t=T_2:\quad \zeta=\zeta_{T_2} \qquad (2.28)$$

$$(\partial V=S=\Sigma_1\cup\Sigma_2\cup S_\infty)$$

The above equations are usual (e. g. see [2]). The surface embedded into water, Σ_1, is determined, i.e., the intersected line $\partial\Sigma_1$ (see Fig. 3) can be determined by the rigid body entering the quiescent water. The value of T_2 is defined at the end of the second stage (the beginning stage of entering water), and is determined. The value of ϕ_{T_1} is determined by the first stage. The value of ζ_{T_2} should be determined by the same relationships of the first stage, i.e. (2.24), according to the similar equation, eq. (2.21) of the first stage, from the following iterative procedure

$$\zeta_{T_2}=-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)|_{z=0,T=T_2}-g^{-1}\rho_0^{-1}p_{a0} \qquad (2.29)$$

We shall prove the existence of the variational solutions of the two-stage problems later. So there is

Property 2 Three iterative procedures for the calculations of the ζ_{T_1} on Σ_2 , the $\overline{(h\Lambda)}$ on $\partial\Sigma_1$, and the ζ_{T_2} , are convergent.

III. Stationary Principles (Generalized Variational Principles)

The sense of the generalized variational principles here is relative to that of the constraint variational principles later, because some constraint conditions must also be satisfied in the generalized variational functionals.

Theorem 1 Setting eqs. (2.11) – (2.16) satisfied a priori from the following functional, the stationary condition of which is the solution of the first stage problem, eqs. (2.1) – (2.16):

$$\begin{aligned}\Pi(\phi,\phi_1,\psi)=&-\int_0^{T_1}\int_V k^{-1}pU(\psi)dVdt-\int_0^{T_1}\int_{\Sigma_1}k^{-1}hp_a dSdt\\&+\int_0^{T_1}\int_{\Sigma_2}k^{-1}p_{a0}\psi dSdt+\int_{\Sigma_1}\rho_{a0}h_0\phi_1|_{t=0}dS-\int_V\rho_0k^{-1}U(\psi_{T_1})\phi|_{t=T_1}dV\\&-\int_0^{T_1}\int_{\partial\Sigma_1}\rho_{a0}\overline{(h\Lambda)}\cdot\boldsymbol{n}\phi_1dc\end{aligned}\qquad(3.1)$$

where $U(\psi)$ is the Heaviside function.

Proof Using the property of Heaviside function, noting eq. (2.19), and employing the following relationships.

$$-\delta(k^{-1}p_a)=-\rho_a^{k-1}\delta\rho_a=\rho_a\delta(\partial\phi_1/\partial t+\Lambda^2/2)\qquad(3.2)$$

we have proved theorem 1:

$$\begin{aligned}\delta\Pi\xlongequal{(2.11)\sim(2.16)}&-\int_0^{T_1}\int_V k^{-1}\rho_0\Delta\phi\delta\phi dVdt+\int_0^{T_1}\int_{\Sigma_1}\Big\{k^{-1}(p_a-p)\delta\psi-\Big[\frac{\partial(\rho_ah)}{\partial t}\\&+\nabla_1\cdot(\rho_ah\Lambda)\Big]\delta\phi_1-k^{-1}\rho_0\Big(\frac{\partial\psi}{\partial t}+\nabla\phi\cdot\nabla\psi\Big)\delta\phi\Big\}dSdt-\int_0^{T_1}\int_{\Sigma_2}k^{-1}\\&\cdot\Big[(p-p_{a0})\delta\psi+\rho_0\Big(\frac{\partial\psi}{\partial t}+\nabla\phi\cdot\nabla\psi\Big)\delta\phi\Big]dSdt+\int_0^{T_1}\int_{S_\infty}\rho_0k^{-1}\boldsymbol{N}\cdot\nabla\phi\delta\phi dSdt\\&+\int_{\Sigma_1}[(\rho_{a0}h_0-\rho_ah)\delta\phi_1|_{t=0}+\rho_ah\delta\phi_1|_{t=T_1}]dS+\int_V\rho_0k^{-1}[U\\&-U(\psi_{T_1})]\delta\phi|_{t=T_1}dV+\int_0^{T_1}\int_{\partial\Sigma_1}[\rho_a h\Lambda-\rho_{a0}\overline{(h\Lambda)}]\cdot\boldsymbol{n}\delta\phi_1dcdt\end{aligned}\qquad(3.3)$$

$$=0\Rightarrow(2.1)\sim(2.10)\qquad(3.4)$$

Theorem 2 Setting eq. (2.27) satisfied a priori from the following functional, the stationary condition of which is the solution of second stage problem, eqs. (2.22) – (2.28):

$$\begin{aligned}\Pi(\phi,\psi)=&-\int_{T_1}^{T_2}\int_V\rho_0^{-1}pU(\psi)dVdt-\int_{T_1}^{T}\int_{\Sigma_1}\boldsymbol{u}\boldsymbol{N}\phi dSdt\\&+\int_{T_1}^{T_2}\int_{\Sigma_2}\rho_0^{-1}p_{a0}\psi dSdt-\int_V U(\psi_{T_2})\phi|_{t=T_2}dV\end{aligned}\qquad(3.5)$$

Proof We have proved theorem 2 from

$$\delta\Pi\overset{(2.27)}{=\!=\!=}-\int_{T_1}^{T_2}\int_V \Delta\phi\delta\phi dVdt-\int_{T_1}^{T_2}\int_{\Sigma_2}\Big[\Big(\frac{\partial\psi}{\partial t}+\nabla\phi\cdot\nabla\psi\Big)\delta\phi$$

$$+\rho_0^{-1}(p-p_{a0})\,\delta\psi\Big]dSdt+\int_{T_1}^{T_2}\int_{\Sigma_1}(\nabla\phi-\boldsymbol{u})\cdot\boldsymbol{N}\delta\phi dSdt$$

$$+\int_{T_1}^{T_2}\int_{S_\infty}\nabla\phi\cdot\boldsymbol{N}\delta\phi dSdt+\int_V[U-U(\psi_{T_2})]\delta\phi|_{t=T_2}dV \tag{3.6}$$

$$=0\Rightarrow(2.22)\sim(2.26),\ (2.28) \tag{3.7}$$

IV. Constraint Variational Principles and Bounded Theorems

Theorem 3 Setting the equilibrium conditions, eqs. (2.3), (2.4) and the continuity condition, eqs. (2.1), (2.2) and (2.5)–(2.10), satisfied a priori from functional Π, eq. (3.1) of the first stage problem respectively, construct functionals Π_1 and $-\Gamma_1$ respectively. Then

(i) Π_1 and Γ_1 have the same property in stationary point:

$$\partial^2\Pi_1=\delta^2\Gamma_1=\int_0^{T_1}\int_V\frac{1}{2}k^{-1}\rho_0|\nabla\delta\phi|^2dVdt+\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}ha^{2m}(1-a^{-2}\Lambda^2)(\delta\Lambda)^2dSdt \tag{4.1}$$

where a is acoustic velocity and $m=(k-1)^{-1}$.

(ii) At least when $\Lambda\leqslant a$, there exists a solution and the following holds

$$\Pi_1\geqslant\Pi_1|_0=-\Gamma_1|_0\geqslant-\Gamma_1 \tag{4.2}$$

where $(\quad)|_0$ is the value in the point of exact solution.

Proof (i) From eq. (3.3) we have proved the stationary condition:

$$\partial\Pi_1=\delta\Pi|_{(2.3)(2.4)} \tag{4.3}$$

(The right-hand side of equality of the above equation indicates that eqs. (2.3), (2.4) are satisfied a priori in $\delta\Pi$ (see eq. (3.3)); It is similar later on)

$$=0\Rightarrow(2.1)(2.2)(2.5)\sim(2.10) \tag{4.4}$$

$$-\delta\Gamma_1=\delta\Pi|_{(2.1)(2.2)(2.5)\sim(2.10)} \tag{4.5}$$

$$=0\Rightarrow(2.3)(2.4) \tag{4.6}$$

(ii) From the conditions, eqs. (2.3), (2.4) and from eqs. (2.11), (2.17), we obtain

$$\Sigma_1:\quad \psi=-g^{-1}\rho_0^{-1}p_a-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)|_{\Sigma_1}-\zeta$$
$$=-g^{-1}\rho_0^{-1}(p_a-p)=0 \tag{4.7}$$

$$\Sigma_2:\quad \psi=-g^{-1}\rho_0^{-1}p_{a0}-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)|_{\Sigma_2}-\zeta$$
$$=-g^{-1}\rho_0^{-1}(p_{a0}-p)=0 \tag{4.8}$$

$$(x,y)\in\Sigma_1:\ \psi=-g^{-1}\rho_0^{-1}p_a-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)|_{z=0}-z \tag{4.9}$$

$$(x,y)\in\Sigma_2:\ \psi=-g^{-1}\rho_0^{-1}p_{a0}-g^{-1}(\partial\phi/\partial t+|\nabla\phi|^2/2)|_{z=0}-z \tag{4.10}$$

eqs. (4.9), (4.10) are continuous on $(x,y)\in\partial\Sigma_1$, and so they are continuous in all domains. We note that eq. (4.8) contains eqs. (2.21) and (2.29) at $t=T_1$ and T_2 respectively employed above.

From eqs. (4.7), (4.8) we know that eqs. (2.3), (2.4) are equivalent to

$$\Sigma:\quad \psi=0 \tag{4.11}$$

and are also equivalent to the following equation when eqs. (2.3), (2.4) are satisfied a priori:

$$\Sigma:\quad \delta\psi=0 \tag{4.12}$$

and simultaneously, the variable ψ in the variational principle, eq. (4.4), should be substituted by eqs. (4.9) and (4.10).

Through the following relationships

$$a^2=1-(2m)^{-1}\Lambda^2,\quad \rho_a=a^{2m} \tag{4.13}$$

we take the variation of eq. (3.2) again:

$$-\delta^2(k^{-1}p_a)=a^{2m}(1-a^{-2}\Lambda^2)(\delta\Lambda)^2/2 \tag{4.14}$$

Taking the variation of eq. (3.1) twice, from eqs. (2.19), (4.12) and (4.14), we obtain the value of $\delta^2\Pi_1$ in eq. (4.1).

Taking the variation of eq. (4.5) again, noting eqs. (2.11) – (2.19), we obtain

$$\begin{aligned}\delta^2\Gamma_1=&\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}k^{-1}\delta(p-p_a)\delta\psi dSdt+\int_0^{T_1}\int_{\Sigma_2}\frac{1}{2}k^{-1}\delta p\delta\psi dSdt\\ =&-\int_0^{T_1}\int_V\frac{1}{2}k^{-1}\delta\left[\rho_0\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz\right)\right]\delta UdVdt-\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}k^{-1}\delta\left\{\rho_a\left[(k\right.\right.\\ &\left.\left.-1)\left(\frac{\partial\phi_1}{\partial t}+\frac{1}{2}\Lambda^2\right)-1\right]\right\}\delta hdSdt\\ =&-\delta\int_0^{T_1}\int_V\frac{1}{2}k^{-1}\rho_0\delta\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz\right)UdVdt-\delta\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}k^{-1}\delta\left\{\rho_a\left[(k\right.\right.\\ &\left.\left.-1)\left(\frac{\partial\phi_1}{\partial t}+\frac{1}{2}\Lambda^2\right)-1\right]\right\}hdSdt+\int_0^{T_1}\int_V k^{-1}\rho_0\delta^2\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz\right)UdVdt\\ &-\int_0^{T_1}\int_{\Sigma_1}k^{-1}\delta^2p_ahdSdt\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}&-\int_0^{T_1}\int_V\frac{1}{2}k^{-1}\rho_0\delta\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz\right)UdVdt\\ &=-\int_V\frac{1}{2}k^{-1}\rho_0(\delta\phi U)|_0^{T_1}dV+\int_0^{T_1}\int_V\frac{1}{2}k^{-1}\rho_0\Delta\phi\delta\phi dVdt\\ &\quad+\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}k^{-1}\rho_0\left(\frac{\partial\psi}{\partial t}+\nabla\phi\cdot\nabla\psi\right)\delta\phi dSdt-\int_0^{T_1}\int_{S_\infty}\nabla\phi\cdot N\delta\phi dSdt \qquad (4.16)\\ &\quad-\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}k^{-1}\delta\left\{\rho_a\left[(k-1)\left(\frac{\partial\phi_1}{\partial t}+\frac{1}{2}\Lambda^2\right)-1\right]\right\}hdSdt\\ &\overset{3.2}{=\!=}-\int_0^{T_1}\int_{\Sigma_1}\frac{1}{2}\rho_ah\delta\left(\frac{\partial\phi_1}{\partial t}+\frac{1}{2}\Lambda^2\right)dSdt\end{aligned}$$

$$=-\int_{\Sigma_1}\frac{1}{2}(\rho_a h\delta\phi_1)|_0^{T_1}dS+\frac{1}{2}\int_0^{T_1}\int_{\Sigma_1}\left[\frac{\partial(\rho_a h)}{\partial t}+\nabla_1\cdot(\rho_a h\Lambda)\right]\delta\phi_1 dSdt$$

$$-\int_0^{T_1}\int_{\partial\Sigma_1}\frac{1}{2}(\rho_a h\Lambda)\cdot\boldsymbol{n}\delta\phi_1 dcdt \tag{4.17}$$

Substituting eqs. (4.14), (4.16) and (4.17) into eq. (4.15), satisfying identical conditions, eqs. (2.1), (2.2) and (2.5) – (2.10), we obtain the value of $\delta^2\Gamma_1$ in eq. (4.1). We have proved eq. (4.1). From the mechanism of constructing functionals Π_1 and Γ_1, we can prove $\Pi_1|_0=-\Gamma_1|_0$ in eq. (4.2) easily. When $\Delta\leqslant a$, we can prove that formula (4.1) is uniformly positive, which is the sufficient condition for functional to be a minimizer, so we have proved eq. (4.2). The equivalence between the minimum principle and existence of the solution is that the harmonic operator is positive in domain (see eq. (2.1)). Up to now, the proof of theorem 3 has been completed.

Theorem 4 Setting the equilibrim condition, eq. (2.24) and the continuity conditions, eqs. (2.22), (2.23), (2.25), (2.26), (2.28), satisfied a priori from functional Π, eq. (3.5) of the second stage problems respectively, construct functionals Π_1 and $-\Gamma_1$ respectively. Then

(i) Π_1 and Γ_1 have the same property in stationary point as follows

$$\delta^2\Pi_1=\delta^2\Gamma_1=\int_{T_1}^{T_2}\int_V\frac{1}{2}|\nabla\delta\phi|^2 dVdt \tag{4.18}$$

(ii) The solution exists and moreover

$$\Pi_1\geqslant\Pi_1|_0=-\Gamma_1|_0\geqslant-\Gamma_1 \tag{4.19}$$

Proof (i) From eq. (3.6) we have proved the stationary condition easily:

$$\delta\Pi_1=\delta\Pi|_{(2.24)} \tag{4.20}$$

$$=0\Rightarrow(2.22)(2.23)(2.25)(2.26)(2.28) \tag{4.21}$$

$$-\delta\Gamma_1=\delta\Pi|_{(2.22)(2.23)(2.25)(2.26)(2.28)} \tag{4.22}$$

$$=0\Rightarrow(2.24) \tag{4.23}$$

(ii) Similarly we can prove that eqs. (4.8), (4.10) (in which the defined domain of eq. (4.10), $(x,y)\in\Sigma_2$, is modified in all regions, and is still indicated by eq. (4.10) later on) are valid in the second stage problem. So eq. (2.24) is equivalent to

$$\Sigma_2:\quad \psi=0 \tag{4.24}$$

and when eq. (2.24) is identical, it is also equivalent to

$$\Sigma_2:\quad \delta\psi=0 \tag{4.25}$$

Simultaneously, variablbe ψ in the variational principle, eq. (4.21), should be substituted by eq. (4.10).

Taking the variation of eq. (3.5) twice, using eq. (4.25), we obtain the value of $\delta^2\Pi_1$ in eq. (4.18).

Take the variation of eq. (4.22) again:

$$\delta^2\Gamma_1=\frac{1}{2}\int_{T_1}^{T_2}\int_{\Sigma_2}\rho_0^{-1}\delta p\delta\psi dSdt=\frac{1}{2}\int_{T_1}^{T_2}\int_V\rho_0^{-1}\delta p\delta UdVdt$$

$$=\frac{1}{2}\delta\int_{T_1}^{T_2}\int_V\rho_0^{-1}\delta pUdVdt-\int_{T_1}^{T_2}\int_V\rho_0^{-1}\delta^2 pUdVdt \tag{4.26}$$

where

$$\frac{1}{2}\int_{T_1}^{T_2}\int_V \rho_0^{-1}\delta p U dV dt = -\frac{1}{2}\int_V (\delta\phi U)\Big|_{t=T_2} dV + \frac{1}{2}\int_{T_1}^{T_2}\int_{\Sigma_2}\Big(\frac{\partial\psi}{\partial t} + \nabla\psi\cdot\nabla\phi\Big)\delta\phi dS dt - \frac{1}{2}\int_{T_1}^{T_2}\int_{\Sigma_1\cup S_\infty}\nabla\phi\cdot N\delta\phi dS dt + \frac{1}{2}\int_{T_1}^{T_2}\int_V \Delta\phi\delta\phi dV dt \tag{4.27}$$

Substituting (4.27) into (4.26), satisfying the identical conditions, eqs. (2.22), (2.23), (2.25), (2.26), (2.28), we obtain the value of $\delta^2\Gamma_1$ in eq. (4.18).

Eq. (4.18) has been proved. We can prove that formula (4.18) is uniformly positive, which satisfies the sufficient condition for the functional to be a minimizer. and know similarly that $\Pi_1|_0 = -\Gamma_1|_0$ in eq. (4.19), so eq. (4.19) holds. The existence of the solution is equivalent to the minimum principle because the harmonic operator is positive. The proof of theorem 4 is completed.

Note 1 Theorem 1 – 4 is the continuity of the work of complementary variational principles in ref. [3], the improvement of which (compared with the classical results, e. g. ref. [4]) is at least that only one determinative formula is required to determine the same property of a pair of complementary functional in stationary point.

V. The Boundary Integral Equation for the Second Stage Problem

Theorem 5 Setting eq. (2.24) satisfied a priori (in which variable ψ should be substituted by variable ϕ from eq. (4.10)), the solution of the second stage problem exists and is determined by

$$\begin{pmatrix} 4\pi \\ 2\pi \end{pmatrix}\int_{T_1}^{T_2}\phi(X_0)dt = \int_{T_1}^{T_2}\int_{\Sigma_2}(\nabla\psi\cdot\nabla r r^{-2}\phi - r^{-1}\partial\psi/\partial t)dS dt - \int_{T_1}^{T_2}\int_{\Sigma_1} N\cdot(\nabla r r^{-2}\phi + u r^{-1})dS dt + \int_V [U(\psi) - U(\psi_{T_2})]r^{-1}\big|_{t=T_2} dV,$$
$$X_0 \in {}^{V}_{S},\quad r = |r - r_0| \tag{5.1}$$

Moreover, it satisfies the relationships of water surface shape and the surface embedded into water as follows

$$\int_{T_1}^{T_2}\int_{\Sigma_2}(\partial\psi/\partial t)dS dt + \int_{T_1}^{T_2}\int_{\Sigma_1} u\cdot N dS dt + \int_V [U(\psi_{T_2}) - U(\psi)]\big|_{t=T_2} dV = 0 \tag{5.2}$$

Proof Taking $\delta\phi = r^{-1}$ and 1 in eq. (4.21), integrating the term of $\int_V \nabla\phi\delta\phi dV$ in eq. (4.21) by part once and twice respectively, we obtain eqs. (5.1) and (5.2) respectively. Because eq. (5.1) is equivalent to (4.21), then from theorem 4 we know that the solution exists. It is eq. (5.2) that is the condition of existence of Neumann problem. The proof of theorem 5 is completed.

VI. The Linearization of Water Surface Integral Region and the Alternative Form of Constraint Variational Principles

By a way similar to property 1, the following holds:

Property 3 When the water interface Σ or Σ'_2 of three-dimensional integral region V and of surface integration in theorems $1-5$, is substituted by $z=0$, the error of the integration is an allowable infinitesimal value.

Note 2 Properties 1 and 2 are capable of solving variables ϕ , ϕ_1 and ψ from variational equation or boundary integral equation once; Property 2 guarantees the convergence of iterative procedure for some undetermined value. So properties $1-3$ provide the capability for calculus of theorems $1-5$.

Variable ψ in constraint variational equation $\delta\Pi_1=0$ of two-stage problems should be substituted by relationships (4.9), (4.10) which are dependent on velocity potential so that the nonlinearity of the variational equation increase. Following corollary 1 provides an alternative way of iteration which is valid obviously for decreasing the nonlinearity of variational equation.

Corollary 1 (i) From eqs. (3.1), (4.12), the variational equation of the first stage problem is equivalent to

$$\begin{aligned}\delta\Pi_1=&-\int_0^{T_1}\int_V k^{-1}\delta pU(\psi)dVdt-\int_0^{T_1}\int_{\Sigma_1}k^{-1}h\delta p_a dSdt\\&+\int_{\Sigma_1}\rho_{a0}h_0\delta\phi_1|_{t=0}dS-\int_V\rho_0 k^{-1}U(\psi_{T_1})\delta\phi|_{t=T_1}dV\\&-\int_0^{T_1}\int_{\partial\Sigma_1}\rho_{a0}\overline{(h\Lambda)}\cdot\boldsymbol{n}\delta\phi_1 dc=0\end{aligned}\tag{6.1}$$

where integral region V and Σ_1 are determined iteratively by eqs. (4.9), (4.10) of the interface function ψ.

(ii) From eqs. (3.5), (4.25), the variational eq. (4.21) of the second stage problem is equivalent to

$$\begin{aligned}\delta\Pi_1=&-\int_{T_1}^{T_2}\int_V\rho_0^{-1}\delta pU(\psi)dVdt-\int_{T_1}^{T_2}\int_{\Sigma_1}\boldsymbol{u}\cdot\boldsymbol{N}\delta\phi dSdt\\&-\int_V U(\psi_{T_2})\delta\phi|_{t=T_2}dV=0\end{aligned}\tag{6.2}$$

where integral region V is determined by eq. (4.10) of interface function ψ .

We note that the treatment in corollary 1 is inverse in note 2: taking the reality (determined by iteration) of water surface boundary, decrease the nonlinearity of variational functional. But the effect of the two is equivalent.

VII. Application in Ship's Wave Resistance Problem

The variational solution of flow past floating body was presented in ref. [5]. For the application in ship's wave resistance problem, the primary effect has been seen in three numerical examples[5]. This problem is the degenerate condition of the second stage problem in this paper: let the stage be steady, and the prescribed velocity u horizontal and constant. As a special application of the second stage problem, the further result of ref. [6] is written here as follows:

Theorem 6[6] The solution of ship's wave resistance of the variational element method and boundary element method exists, and is unique, the accuracy of which achieves the engineering requirement.

Otherwise the following holds obviously.

Corollary 2 The second stage problem provides a rigorous theoretical basis of the variational finite element method and boundary element method for calculating the unsteady rigid body-water coupled problem, containing ship's unsteady wave-making problem.

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