

CRITERIA FOR FINITE ELEMENT ALGORITHM OF GENERALIZED HEAT CONDUCTION EQUATION

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Abstract

To eliminate oscillation and overbounding of finite element solutions of classical heat conduction equation, the author and Xiao have put forward two new concepts of monotonicities and have derived and proved several criteria. This idea is borrowed here to deal with generalized heat conduction equation and finite element criteria for eliminating oscillation and overbounding are also presented. Some new and useful conclusions are drawn.

Key words heat conduction equation, hyperbolic differential equation, finite element method, criteria, oscillation, overbounding

I. Introduction

The classical heat conduction equation commonly referred to is a parabolic differential equation, whose finite element solutions are very likely to display oscillation and/or overbounding (called excessiveness in the authors previous papers). To eliminate them, the author et al. put forward the two new concepts of time monotony and spatial monotony of finite element solutions and derived and proved several criteria to guarantee reasonable numerical solutions^[1-3], which is viewed as a major progress in the field.

Classical (parabolic) heat conduction equation is

$$\left. \begin{aligned} \frac{1}{\alpha} \frac{\partial T}{\partial t} &= \nabla^2 T(\mathbf{x}, t) + f(\mathbf{x}, t), \quad T(\mathbf{x}, 0) = T_0, \quad \mathbf{x} \in V \\ T &= T_1, \quad \mathbf{x} \in S_1; \quad \frac{\partial T}{\partial n} = \rho, \quad \mathbf{x} \in S_2; \quad \frac{\partial T}{\partial n} = \beta(T_3 - T), \quad \mathbf{x} \in S_3 \end{aligned} \right\} \quad (1.1)$$

Some people have pointed out the defect of classical heat conduction equation that a perturbation will propagate to every point of the whole domain V at the very instant when it occurs. That is to say, the propagating speed of heat is infinite^[4], which is obviously unreasonable. Therefore some people added a new term into Eq. (1.1) representing the effect of the speed. This turned the parabolic equation into a hyperbolic one^[4], which is named generalized heat conduction equation in this paper. It is written as^[5]

$$\left. \begin{aligned} \frac{1}{v^2} \frac{\partial^2 T}{\partial t^2} + \frac{1}{\alpha} \frac{\partial T}{\partial t} &= \nabla^2 T(\mathbf{x}, t) + f(\mathbf{x}, t) \\ T(\mathbf{x}, 0) &= T_0, \quad \partial T / \partial t|_{t=0} = 0, \quad \mathbf{x} \in V \\ T &= T_1, \quad \mathbf{x} \in S_1; \quad \frac{\partial T}{\partial n} = \rho, \quad \mathbf{x} \in S_2; \quad \frac{\partial T}{\partial n} = \beta(T_3 - T), \quad \mathbf{x} \in S_3 \end{aligned} \right\} \quad (1.2)$$

When the speed V approaches the infinite, the generalized heat conduction equation degenerates into a classical one.

The appearance of the second-order derivative of T to t in Eq. (1.2) results in dramatic changes in Eq. (1.1). Since Eq. (1.2) is usually used to describe the phenomena of wave propagation, vibration and so on, its finite element solutions are much more likely to show oscillation and hence the problem of eliminating oscillation can be more complicated.

The numerical solutions of transient heat conduction should be compatible with the natural law of heat conduction whether the governing equation is a parabolic one or a hyperbolic one. That is to say, heat always transfers from the warmer part to the cooler part. Therefore the concepts of monotonicities put forward in [1-3] are also useful in dealing with the generalized heat conduction equation.

II. Finite Element Formulation and Concept of Monotonies

Discretize V into a number of small elements V_i . Assume element shape function matrix as $[N(x)]$ and nodal temperature vector $q(t)$. Then

$$T(x, t) = \sum_i [N(x)] q_i(t), \quad \frac{\partial T}{\partial t} = \sum_i [N(x)] \dot{q}_i(t), \quad \frac{\partial^2 T}{\partial t^2} = \sum_i [N(x)] \ddot{q}_i(t).$$

Eq. (2.1) can be derived from Eq. (1.2) by the Galerkin residual method

$$\left. \begin{aligned} [M] \ddot{q}(t) + [C] \dot{q}(t) + [K] q(t) &= F \\ [M] &= \frac{1}{v^2} \sum_i \iiint_{V_i} [N]^T [N] dV, \quad [C] = \frac{1}{\alpha} \sum_i \iiint_{V_i} [N]^T [N] dV \\ [K] &= [K_v] + [K_s]; \quad [K_v] = \sum_i \iiint_{V_i} [\nabla N]^T [\nabla N] dV \\ [K_s] &= \sum_i \iint_{S_k} \beta [N]^T [N] dS \\ F &= \sum_i \iiint_{V_i} f [N]^T dV + \sum_i \iint_{S_j} \rho [N]^T dS + \sum_i \iint_{S_k} \beta T_s [N]^T dS \end{aligned} \right\} \quad (2.1)$$

where S_j represents the external boundary of j th element on S_2 and S_k the external boundary of the k th element of S_3 . To the l th node on S_1 , modify K_{ll} as BK_{ll} and F_l as $BK_{ll}T_1$, where B is a great number, say $B = 10^6$.

Further discretization of the first equation of Eq. (2.1) on time interval $[t_{n-1}, t_{n+1}]$ in time dimension gives

$$\left. \begin{aligned} [A] q^{n+1} &= [B] q^n + [D] q^{n-1} + F \Delta t^2 \\ F &= \sum_i \iint_{S_j} \rho [N]^T dS + \sum_i \iint_{S_k} \beta T_s [N]^T dS \\ [A] &= [M] + \theta_1 \Delta t [C] + \theta_2 \Delta t^2 [K] \\ [B] &= 2[M] + (2\theta_1 - 1) \Delta t [C] + (2\theta_2 - \theta_1 - 1/2) \Delta t^2 [K] \\ [D] &= -[M] + (1 - \theta_1) \Delta t [C] + (\theta_1 - \theta_2 - 1/2) \Delta t^2 [K] \end{aligned} \right\} \quad (2.2)$$

where θ_1 and θ_2 are factors of numerical integration. $\theta_1 = 1.5$ and $\theta_2 = 1$ are called backward-difference schemes and $\theta_1 = 0.5$ and $\theta_2 = 0.25$ are called average-acceleration schemes.

Taking initial value T_0 as a constant, $q^0 = T_0 I$ where $I^T = \{1, 1, \dots, 1\}$. Taylor

expansion of q at t_1 gives

$$q^1 = q^0 + \dot{q}^0 \Delta t + \ddot{q}^0 \Delta t^2 / 2 \quad (2.3)$$

From the initial condition $\partial T / \partial t|_{t=0} = 0$ that is $\dot{q}^0 = 0$, and hence $\ddot{q}^0 = [M]^{-1} (F - [K]q^0)$. Therefore the initial condition of the recurrence formula—Eq. (2.2) becomes

$$q^0 = T_0 I; \quad q^1 = q^0 + \Delta t^2 [M]^{-1} (F - [K]q^0) / 2 \quad (2.4)$$

In heat conduction, heat always transfers from the warmer part to the cooler part. Consequently, the temperature at any point will increase monotonically if $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$ (heat flows into V); the temperature at any point will decrease monotonically if $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$ (heat flows out of V). In a word, the solutions of passive heat conduction equation should possess time monotony. Furthermore, the solutions of some special boundary-value problems of passive heat conduction should possess spatial monotony, too.

III. Time Monotony

Definition At any time t_n , $q^{n+1} \geq q^n$ if $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$; or $q^{n+1} \leq q^n$ if $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$.

Lemma 1 $F - [K]q^0 \geq 0$ if $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$; or $F - [K]q^0 \leq 0$ if $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$.

Proof To the cases with the first boundary and third boundary, Lemma 1 in fact was proved in [1]. So here only the second boundary is to be investigated. To node j on the second boundary, its corresponding row of 'stiffness' matrix $[K]$ need not be changed but its corresponding element on the right term becomes

$$(F - [K]q^0)_j = \iint_{S_j} \rho N_j^T dS - \sum_{i=1}^m K_{ji} T_i = \iint_{S_j} \rho N_j^T dS \quad (3.1)$$

where m is the number of the nodes. Let us consider a special family of linear elements (called as C^1 family elements in this paper): 1-D linear element, plane triangular element, four-node isoparameter element, tetrahedron element, spatial eight-node isoparameter element. On the boundary of any C^1 family element, $N_j^T \geq 0$. Accordingly, Lemma 1 holds for the second boundary.

Lemma 2 If Eq. (3.2) is satisfied, then $q^2 \geq q^1$ when $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$; or $q^2 \leq q^1$ when $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$.

$$[A]^{-1}([E] + ([B] - [A])[M]^{-1}/2) \geq 0 \quad (3.2)$$

where $[E]$ is a unit matrix.

Proof From Eq. (2.2)

$$\begin{aligned} [A](q^2 - q^1) &= ([B] - [A])q^1 + [D]q^0 + F\Delta t^2 \\ &= ([B] - [A])(q^1 - q^0) + ([D] + [B] - [A])q^0 + F\Delta t^2 \end{aligned} \quad (3.3)$$

As $[D] + [B] - [A] = -[K]\Delta t^2$ and $q^1 - q^0 = \Delta t^2 [M]^{-1} (F - [K]q^0) / 2$, Eq. (3.3) becomes

$$\begin{aligned} [A](q^2 - q^1) &= ([B] - [A])(q^1 - q^0) + (F - [K]q^0)\Delta t^2 \\ &= \Delta t^2 ([E] + ([B] - [A])[M]^{-1}/2)(F - [K]q^0) \end{aligned} \quad (3.4)$$

Examination of Eq. (3.4) shows that if Eq. (3.2) is satisfied, Lemma 2 is justified according to Lemma 1.

Theorem 1 If Eqs. (3.2) and (3.5) are satisfied, then at any t_n , $q^{n+1} \geq q^n$ when $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$; or $q^{n+1} \leq q^n$ when $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$.

$$[A]^{-1}[B] \geq 0; [A]^{-1}[D] \geq 0; [M]^{-1} \geq 0 \quad (3.5)$$

Proof According to Lemma 1 and Lemma 2, if Eq. (3.2) is satisfied and $[M]^{-1} \geq 0$, $q^1 \geq q^0$ and $q^2 \geq q^1$ when $\rho \geq 0$ and $T_1 \geq T_0$ and $T_3 \geq T_0$; or $q^1 \leq q^0$ and $q^2 \leq q^1$ when $\rho \leq 0$ and $T_1 \leq T_0$ and $T_3 \leq T_0$. Another recurrence formula can be derived from Eq. (2.2) as

$$[A](q^{n+1} - q^n) = [B](q^n - q^{n-1}) + [D](q^{n-1} - q^{n-2}) \quad (3.6)$$

Thus it is obvious that satisfaction of $[A]^{-1}[B] \geq 0$ and $[A]^{-1}[D] \geq 0$ will make $q^{n+1} \geq q^n$ if $q^{n-1} \geq q^{n-2}$ and $q^n \geq q^{n-1}$; or $q^{n+1} \leq q^n$ if $q^{n-1} \leq q^{n-2}$ and $q^n \leq q^{n-1}$. According to the recurrence relation between q^{n+1} and q^n and inductive method, Theorem 1 is justified.

Theorem 2 Eq. (3.7) is one group of sufficient conditions for finite element solutions without oscillation, which is what the author has referred to as the criterion of time monotony.

$$\left. \begin{aligned} [M]^{-1} \geq 0; [A]^{-1}([E] + ([B] - [A])[M]^{-1}/2) \geq 0 \\ [A]^{-1}[B] \geq 0; [A]^{-1}[D] \geq 0 \end{aligned} \right\} \quad (3.7)$$

Theorem 2 is an inference of Theorem 1, which is needless to verify.

There are other criteria that can be derived from Eq. (3.7).

Theorem 3 Eq. (3.8) is another criterion of time monotony.

$$[M]^{-1} \geq 0; [A]^{-1} \geq 0; [E] + ([B] - [A])[M]^{-1}/2 \geq 0; [B] \geq 0; [D] \geq 0 \quad (3.8)$$

As known before, finite element solutions in better precision of the classical heat conduction equation and dynamic equation would result from suitable technique of 'mass' lumping^[6]. So 'mass' is lumped in $[M]$ and $[C]$, which makes $[M]^{-1} \geq 0$.

The appearance of $[A]^{-1}$ in Eqs. (3.7) and (3.8) makes it inconvenient to use them. Fortunately, there exists a useful conclusion in matrix theory. It says: If all the off-diagonal elements of a positive-definite matrix are non-positive, then the inverse of the matrix must be a non-negative one^[7]. After mass lumping, M_{ij} and C_{ij} ($i \neq j$) become zero, $K_{ij} \leq 0$ is enough to make $A_{ij} \leq 0$ ($i \neq j$). It can be proved that $K_{ij} \leq 0$ ($i \neq j$) for 1-D linear element, plane triangular element and tetrahedron element with internal angles not greater than $\pi/2$. Further investigation of Eq. (2.2) gives Theorem 4.

Theorem 4 Using the above-mentioned elements, Eqs. (3.9) and (3.10) are another group of sufficient conditions for time monotony.

$$2\theta_2 - \theta_1 - 1/2 \leq 0; \theta_1 - \theta_2 - 1/2 \leq 0; \theta_2 - \theta_1 - 1/2 \leq 0 \quad (3.9)$$

$$\left. \begin{aligned} 2M_{ii} + (2\theta_1 - 1)\Delta t C_{ii} + (2\theta_2 - \theta_1 - 1/2)\Delta t^2 K_{ii} &\geq 0 \\ -M_{ij} + (1 - \theta_1)\Delta t C_{ij} + (\theta_1 - \theta_2 - 1/2)\Delta t^2 K_{ij} &\geq 0 \\ 3M_{ii} + (\theta_1 - 1)\Delta t C_{ii} + (\theta_2 - \theta_1 - 1/2)\Delta t^2 K_{ii} &\geq 0 \end{aligned} \right\} \quad (3.10)$$

Proof Satisfaction of Eqs. (3.9) and (3.10) will lead to satisfaction of Eq. (3.8).

Eq. (3.9) is a requirement to the type of the schemes used. And Eq. (3.10) serves as a restriction upon element type, element size and time step length.

IV. Discussion of Different Schemes

If $\theta_1, \theta_2 \geq 0$, only the first two equations in Eq. (3.9) are to be tested and $\theta_1 \leq 1.5$ and $\theta_2 \leq 1$ can be derived. Let us discuss the validity of the schemes in term of conditional stability and absolute stability in the following.

(1) Absolutely stable schemes

Absolute stability requires Eq. (2.2) to satisfy

$$\theta_1 \geq 0.5, \theta_2 \geq (\theta_1 + 1/2)^2/4 \quad (4.1)$$

Combined with Eq. (3.9), it can be seen that those schemes that satisfy Eq. (4.2) and at the same time are likely to eliminate oscillation are

$$0.5 \leq \theta_1 \leq 1.5, 0.25 \leq \theta_2 \leq 1, 2\theta_2 - \theta_1 \leq 0.5, \theta_1 - \theta_2 \leq 0.5, \theta_2 \geq (\theta_1 + 1/2)^2/4 \quad (4.2)$$

Consequently, the average-acceleration scheme and backward-difference scheme satisfy Eq. (4.2). But as the scheme with $\theta_1 = 1.5$ and $\theta_2 = 1$ does not satisfy the second inequality in Eq. (3.10), only the absolutely stable average-acceleration scheme among the two is likely to eliminate oscillation. To this scheme, Eq. (3.10) becomes

$$\left. \begin{aligned} 2M_{ii} - 0.5\Delta t^2 K_{ii} &\geq 0, \quad -M_{ii} + 0.5\Delta t C_{ii} - 0.25\Delta t^2 K_{ii} \geq 0 \\ 3M_{ii} - 0.5\Delta t C_{ii} - 0.75\Delta t^2 K_{ii} &\geq 0 \end{aligned} \right\} \quad (4.3)$$

Since satisfaction of the third expression of Eq. (4.3) will lead to the first one, only the last two ones are to be preserved. Denote $r_i = M_{ii}/K_{ii}$, $s_i = C_{ii}/K_{ii}$, thus Eq. (4.3) is equivalent to Eq. (4.4)

$$\left. \begin{aligned} \max_i (s_i - \sqrt{s_i^2 - 4r_i}) &\leq \Delta t \leq \min_i [s_i + \sqrt{s_i^2 - 4r_i}] \\ (\sqrt{s_i^2 + 36r_i} - s_i)/3 &\leq \Delta t \leq \min_i [(\sqrt{s_i^2 + 48r_i} - s_i)/4] \end{aligned} \right\} \quad (4.4)$$

In Eq. (4.4), the first one puts a restriction between time step length and element used while the second a requirement to the size, type and mesh of the element.

(2) Conditionally stable scheme

To the central-difference scheme, a conditionally stable scheme of $\theta_1 = 0.5$ and $\theta_2 = 0$, stability requires $\Delta t < 2/\max \omega_i^{(0)}$, $\omega_i = \sqrt{K_i/M_i}$, where K_i and M_i are the i th diagonal elements of the diagonal matrices of $[K]$ and $[M]$ respectively through similarity transformation so that ω_i is like the natural frequency in dynamics. To this scheme, Eq. (3.10) becomes

$$2M_{ii} - \Delta t^2 K_{ii} \geq 0, 0.5\Delta t C_{ii} - M_{ii} \geq 0, 3M_{ii} - 0.5\Delta t C_{ii} - \Delta t^2 K_{ii} \geq 0 \quad (4.5)$$

It can be compactly written as

$$2M_{ii} - \Delta t^2 K_{ii} \geq 0.5\Delta t C_{ii} - M_{ii} \geq 0 \quad (4.6)$$

Combined with the stability requirement, the lower and upper bounds for Δt are obtained

$$\max_i \frac{2r_i}{s_i} \leq \Delta t \leq \min_i [\sqrt{2r_i}, (\sqrt{s_i^2 + 48r_i} - s_i)/4], \Delta t < \min_i \frac{2}{\omega_i} \quad (4.7)$$

Accordingly, there exists no limitation to element size (but there is restriction between element size and time step length). But the scheme is less convenient in a sense for the spectral radius must be determined before the upper bound.

To other schemes, wise estimates for Δt should come from the most general criteria, Eqs. (3.9) and (3.10), without direct lower and upper bound formulas like Eq. (4.4) or Eq. (4.7).

V. Explicit Lower and Upper Bound Formulas

Consider a 1-D heat conduction on a rod: its left end is adiabatic (the second boundary condition of $\rho=0$) and its right is convective heat-transferred (the third boundary condition) or temperature-specified (the first boundary condition). Linear elements of size Δx and mass lumping are used. This results in

$$[M]=[G]/v^2, [C]=[G]/\alpha, F=\{0, 0, \dots, 0, \beta T_s\}^T$$

$$[G]=\frac{\Delta x}{2} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & \ddots & \\ 0 & & & & 2 \\ & & & & & 1 \end{bmatrix}, [K]=\frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & -1 & 2 & -1 \\ & & & & -1 & 1+\beta\Delta x \end{bmatrix} \quad (5.1)$$

(1) Average-acceleration scheme

$$1 - \sqrt{1 - 8\alpha^2/(\nu\Delta x)^2} \leq 2\alpha\Delta t/\Delta x^2$$

$$\leq \frac{1}{1+\beta\Delta x} \min[1 + \sqrt{1 - 8\alpha^2(1+\beta\Delta x)/(\nu\Delta x)^2},$$

$$(\sqrt{1 + 72\alpha^2(1+\beta\Delta x)/(\nu\Delta x)^2} - 1)/3] \quad (\text{for } S_3) \quad (5.2)$$

$$1 - \sqrt{1 - 8\alpha^2/(\nu\Delta x)^2} \leq 2\alpha\Delta t/\Delta x^2 \leq \min[1 + \sqrt{1 - 8\alpha^2/(\nu\Delta x)^2},$$

$$(\sqrt{1 + 72\alpha^2/(\nu\Delta x)^2} - 1)/3] \quad (\text{for } S_1) \quad (5.3)$$

For any boundary conditions at right end, $\Delta x \geq 2\alpha^2(2\beta + \sqrt{4\beta + 2\nu^2/\alpha^2})/\nu^2$ should be satisfied.

(2) Central-difference scheme

As pointed out, the spectral radius must be determined before the upper bound for Δt can be decided. The eigenvalues of a large matrix are hard to evaluate accurate enough and there is no versatile formula for this purpose. But for the maximum eigenvalue, fortunately, Irons has proved: the maximum eigenvalue of the system (total stiffness matrix) is not greater than that of its elements^[6]. Consequently, $\max_i \omega_i^2 \leq \nu^2/\Delta x^2(2 + \beta\Delta x + \sqrt{4 + (\beta\Delta x)^2})$ for convective boundary and $\max_i \omega_i^2 \leq \nu^2/4\Delta x^2$ for temperature-specified boundary. Thus

$$\frac{2\alpha}{\nu^2} \leq \Delta t \leq \frac{1}{1+\beta\Delta x} \min\left[\frac{\Delta x \sqrt{1+\beta\Delta x}}{\nu}, \frac{\Delta x^2}{8\alpha} \left(\sqrt{1 + \frac{48\alpha^2(1+\beta\Delta x)}{(\nu\Delta x)^2}} - 1\right)\right]$$

$$\Delta t < \Delta x \sqrt{2 + \beta\Delta x + \sqrt{4 + (\beta\Delta x)^2}}/\nu \quad (\text{for } S_3) \quad (5.4)$$

$$\frac{2\alpha}{\nu^2} \leq \Delta t \leq \min\left[\frac{\Delta x}{\nu}, \frac{\Delta x^2}{8\alpha} \left(\sqrt{1 + \frac{48\alpha^2}{(\nu\Delta x)^2}} - 1\right)\right] \quad (\text{for } S_1) \quad (5.5)$$

VI. Spatial Monotony

Definition At any t_n , $q_i^n \geq q_{i-1}^n$ if $q_i^n \geq q_0^n$ or $q_i^n \leq q_{i-1}^n$ if $q_i^n \leq q_0^n$ where

$i = 1, 2, \dots, m$.

As mentioned before, the temperature fields of passive heat conduction should always possess time monotony (with time-independent boundary conditions). Furthermore, some special boundary-value problems should still have spatial monotony. The example presented in Section V is just a simple one of them — the temperature distribution in space is a monotonic curve form left to right.

Unlike time monotony, general conclusions on spatial monotony can not be obtained so easily because of the failure to derive the explicit expression of $q_i^n - q_{i-1}^n$. But for the 1-D example just mentioned, the author has got some useful results fortunately.

Lemma 3 For a 1-D heat conduction whose left end is adiabatic and whose right is convective heat-transferred or temperature-specified, when linear elements of size Δx are used, and then at any t_n , the temperature field is a monotonically decreasing curve from left to right if $[K]q^n \geq 0$ and vice versa if $[K]q^n \leq 0$. Note here the last element of $[K]q^n$ is not included in the present discussion of $[K]q^n \geq 0$ or $[K]q^n \leq 0$.

Proof the product vector of $\Delta x[K]q^n$ is

$$\Delta x \cdot [K]q^n = \begin{Bmatrix} q_0^n - q_1^n \\ -q_0^n + 2q_1^n - q_2^n \\ -q_1^n + 2q_2^n - q_3^n \\ \vdots \\ -q_{m-2}^n + 2q_{m-1}^n - q_m^n \\ -q_{m-1}^n + (1 + \beta \Delta x)q_m^n \end{Bmatrix} = \begin{Bmatrix} q_0^n - q_1^n \\ q_1^n - q_2^n - (q_0^n - q_1^n) \\ q_2^n - q_3^n - (q_1^n - q_2^n) \\ \vdots \\ q_{m-1}^n - q_m^n - (q_{m-2}^n - q_{m-1}^n) \\ -q_{m-1}^n + (1 + \beta \Delta x)q_m^n \end{Bmatrix} \quad (6.1)$$

From Eq. (6.1), there exists a recurrence relation between any two adjacent elements of $[K]q^n$ except the last element so that if $[K]q^n \geq 0$, that is, $q_0^n - q_1^n \geq 0$, $q_1^n - q_2^n \geq q_0^n - q_1^n \geq 0$, $q_2^n - q_3^n \geq q_1^n - q_2^n \geq 0$, ..., $q_{m-1}^n - q_m^n \geq q_{m-2}^n - q_{m-1}^n \geq \dots \geq q_0^n - q_1^n \geq 0$, then the temperature field is just like a monotonically decreasing curve from left to right. For the same reason, Lemma 3 holds when $[K]q^n \leq 0$ too.

Theorem 5 For the above-mentioned example, if the solution has time monotony and satisfies Eq. (6.2), then it also has spatial monotony.

$$[B] \geq 0; [D] \geq 0 \quad (6.2)$$

Proof Rewrite the first formula of Eq. (2.2) as

$$\Delta t^2 [K]q^{n+1} = F \Delta t^2 + [B](q^n - q^{n+1}) + [D](q^{n-1} - q^{n+1}) \quad (6.3)$$

When the solution possesses time monotony, that is $q^n \geq q^{n+1}$ or $q^n \leq q^{n+1}$ at any t_n , and because $[B] \geq 0$ and $[D] \geq 0$ and all the elements of F except the last one are zero, $[K]q^{n+1} \geq 0$ or $[K]q^{n+1} \leq 0$ except the last element of $[K]q^{n+1}$ at any t_n . And according to Lemma 3, the solution has spatial monotony.

Theorem 6 If we satisfy Eq. (3.8), or Eqs. (3.9) and (3.10), or Eq. (4.4), or Eq. (4.7), or Eqs. (5.2) and (5.3), or Eqs. (5.4) and (5.5), then the solution possesses both time monotony and spatial monotony.

Proof The conclusions on time monotony in Theorem 6 have been proved before. If any one of the above-mentioned criteria is satisfied, it means that $[B] \geq 0$ and $[D] \geq 0$ so that Theorem 6 holds according to Theorem 5.

From Theorem 6, it can be concluded that if the solution has time monotony, then it is very likely for it to have spatial monotony in most cases. But the inverse is not true, which will be revealed in Section VII.

VII. Numerical Example and Analysis

Consider the above-mentioned example: Let $\alpha=20$, $\beta=0.03$, $\Delta x=10$, $v=10$, $T_0=100$, $T_s=50$. Discretize V into thirty linear elements. Average-acceleration scheme is used. Different values of Δt are used in computation to justify the conclusions obtained. Due to the limitation of the space of the paper, the results at only the first four steps are listed in the following tables. From Eq. (5.2), $0.44 \leq \Delta t \leq 0.76$.

(1) When Δt satisfying the inequality is used, the solution has both time monotony and spatial monotony, which justifies those conclusions. The results of $\Delta t=0.5$ are shown in Table 1

Table 1 Results of $\Delta t=0.5$

$t \backslash x$	250	260	270	280	290	300
0.5	100.00	100.00	100.00	100.00	100.00	98.25
1.0	100.00	100.00	100.00	100.00	99.82	93.93
1.5	100.00	100.00	100.00	99.99	99.53	91.90
2.0	100.00	100.00	100.00	99.98	99.17	90.18

(2) If Δt greater than the upper bound 0.76 up to 0.85 is taken, still, monotonic solution would be obtained since the criteria are sufficient conditions. But for the sake of safety, Δt between the lower bound and upper bound is advisable. Even though it is hard to satisfy this requirement, Δt should be as close to any one of the two bounds as possible. The results of $\Delta t=0.8$ are listed in Table 2.

Table 2 Results of $\Delta t=0.8$

$t \backslash x$	250	260	270	280	290	300
0.8	100.00	100.00	100.00	100.00	100.00	90.40
1.6	100.00	100.00	100.00	99.98	99.25	89.88
2.4	100.00	100.00	100.00	99.91	98.51	87.06
3.2	100.00	100.00	99.99	99.80	97.71	85.53

(3) If Δt greater than 0.85 up to 1.92 is taken, the solution has only spatial monotony without time monotony, which means that time monotony and spatial monotony are different. The results of $\Delta t=1$ are shown in Table 3, in which the solution oscillates at $x=300$ (right end).

Table 3 Results of $\Delta t=1$

$t \backslash x$	250	260	270	280	290	300
1.0	100.00	100.00	100.00	100.00	100.00	85.00
2.0	100.00	100.00	100.00	99.95	98.59	87.60
3.0	100.00	100.00	99.99	99.81	97.56	83.92
4.0	100.00	100.00	99.97	99.58	96.48	83.24

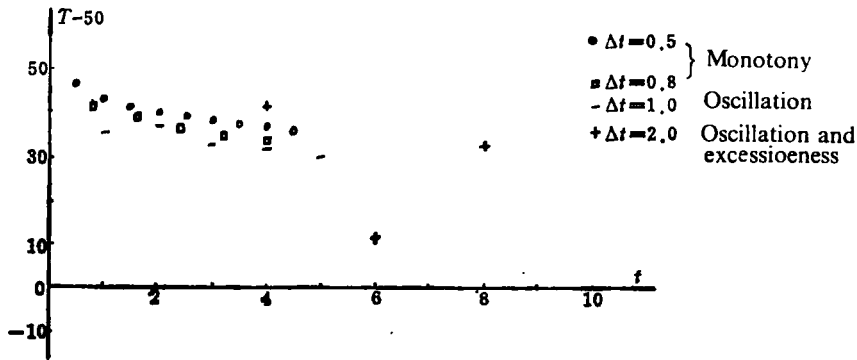
(4) If still greater Δt is taken, the solution displays no monotonies. The results of $\Delta t=2$ are shown in Table 4. It should be pointed out that oscillation usually diminishes as time steps forward and might vanish finally.

Table 4 Results of $\Delta t = 2$

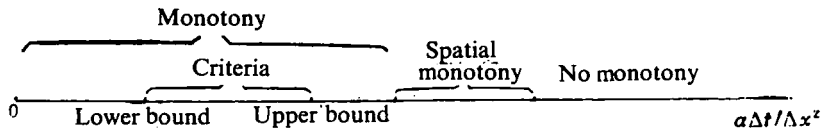
$\frac{t}{x}$	250	260	270	280	290	300
2.0	100.00	100.00	100.00	100.00	100.00	40.00
4.0	100.00	100.00	99.95	99.34	90.84	92.48
6.0	100.00	99.98	99.77	98.08	91.76	62.96
8.0	99.99	99.92	99.43	96.99	88.48	82.80

(5) If Δt less than the lower bound 0.44 is taken, the monotonic solution would be obtained. It seems to mean that due to mass-lumping, the lower bound might not affect the solution, so is the solution of parabolic heat conduction equation when mass is lumped^[2, 3]. But it is rather doubtful whether this conclusion holds for all stable schemes or not in solving hyperbolic heat conduction equation.

According to the numerical results, the temperature at the right end varying with time can be shown in Fig. 1.

**Fig. 1** Temperature at the right end with different Δt

The above-mentioned five kinds of phenomena resulting from different Δt are similar to those of classical heat conduction equation. Accordingly, a similar distribution of monotones is shown in Fig. 2.

**Fig. 2** Monotony distribution along the axis of $a\Delta t/\Delta x^2$ with lumped mass

VIII. Conclusions

(1) Under certain conditions, the finite element solutions of hyperbolic heat conduction equation can be non-oscillating, that is to say, it has time monotony.

(2) The author has derived and proved several criteria for eliminating oscillation.

(3) For average-acceleration scheme (absolutely stable) and central-difference scheme (conditionally stable), the lower and upper bound formulas for Δt have been given.

(4) For 1-D problems with linear elements, explicit lower and upper bound formulas for Δt are presented.

(5) Spatial monotony is also discussed.

(6) Distribution of monotones on the axis $\alpha \Delta t / \Delta x^2$ for 1-D problems is given finally.

(7) The criteria of the hyperbolic heat conduction equation are much more complicated than those of the parabolic one, and the region with monotones is smaller. If oscillation occurs, its amplitude will be greater.

In general, the idea in solving classical heat conduction equation is borrowed in the paper to deal with the generalized one and new conclusions have been drawn.

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