

THE RANDOM VARIATIONAL PRINCIPLE AND FINITE ELEMENT METHOD

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Abstract

In this paper, we introduced the random materials, geometrical shapes, force and displacement boundary condition directly into the functional variational formulations and developed a unified random variational principle and finite element method with the small parameter perturbation method. Numerical examples showed that the methods have the advantages of the simple and convenient program implementation, and are effective for the random mechanics problems.

Key words variational principle, finite element method, perturbation method

I. Introduction

In the mechanics analysis of different kind of engineering structures, it is necessary for many problems to be analysed deterministically. The method widely used is the finite element method based on the variational principles. However, some engineering structures often have the random materials, geometrical shapes, force and displacement boundary conditions. It is obvious that the deterministic analysing method is not used completely for these structures. Hence, how the relevant variational principles and finite element methods are found according to characteristics of random variables is to construct random variational principles and stochastic finite element method for dealing with random mechanics problems.

Some stochastic finite element methods^[1, 3, 4] have been developed in the recent ten years, but most of the methods are formulated in the framework of the direct stiffness matrix approaches, so they are unable to consistently and unifyly incorporate the random materials, geometrical shapes, force and displacement boundary conditions into the finite element formulations, to construct multivariable finite element modals and to develop new and effective numerical analysing methods. Therefore, developing various random variational principles has theoretical significance and practical values. Based on the deterministic variational principle (i.e. conventional variational principle), a random variational principle and stochastic finite element method have been developed in this paper by using second-order perturbation techniques according to the random characteristics.

II. The Random Variational Principle

For an elastic body with small deformation, the minimum potential energy variational principle is

$$\int_{\Omega} \delta \mathbf{e}^T \mathbf{D} \mathbf{e} d\Omega - \int_{\Omega} \delta \mathbf{u}^T \bar{\mathbf{f}} d\Omega - \int_{\Gamma} \delta \mathbf{u}^T \bar{\mathbf{p}} d\Gamma = 0 \quad (2.1)$$

The prescribed traction and displacement are given by

$$\begin{aligned}\sigma^T n &= \bar{p} & (\text{on } \Gamma_p) \\ u &= \bar{u} & (\text{on } \Gamma_u)\end{aligned}\quad (2.2)$$

The random variation of material properties, shape, force and displacement prescribed boundary conditions can be regarded as spatial random field $b(x)$, where x is spatial coordinates.

The displacement random function u is expanded as a Taylor series about the mean of the random field $b(x)$, denoted by $\bar{b}(x)$. We have

$$u(b(x), x) \doteq u(\bar{b}(x), x) + \left(\frac{\partial u}{\partial b}\right)_{\bar{b}} db + \frac{1}{2} \left(\frac{\partial^2 u}{\partial b^2}\right)_{\bar{b}} (db)^2 + \dots \quad (2.3a)$$

Now let $db = (b - \bar{b})\varepsilon$ and equation (2.3a) become

$$u(b(x), x) \doteq u(\bar{b}(x), x) + \left(\frac{\partial u}{\partial b}\right)_{\bar{b}} (b - \bar{b})\varepsilon + \frac{1}{2} \left(\frac{\partial^2 u}{\partial b^2}\right)_{\bar{b}} (b - \bar{b})^2 \varepsilon^2 + \dots \quad (2.3b)$$

Via a second-order perturbation at a given x equation (2.3b) may be written as

$$u \doteq u^0 + \varepsilon u' + \varepsilon^2 u'' \quad (2.3)$$

where the superscripts (0), (') and (") are the random function u , the first-order variation and the second-order variation evaluated at b , respectively; ε is a given small parameter ($\varepsilon = db/(b - \bar{b})$).

For the random functions e , D , \bar{f} , \bar{p} , \bar{u} , and n , the expansion is done similarly by

$$e \doteq e^0 + \varepsilon e' + \varepsilon^2 e'' \quad (2.4)$$

$$D \doteq D^0 + \varepsilon D' + \varepsilon^2 D'' \quad (2.5)$$

$$\bar{f} \doteq \bar{f}^0 + \varepsilon \bar{f}' + \varepsilon^2 \bar{f}'' \quad (2.6)$$

$$\bar{p} \doteq \bar{p}^0 + \varepsilon \bar{p}' + \varepsilon^2 \bar{p}'' \quad (2.7)$$

$$\bar{u} \doteq \bar{u}^0 + \varepsilon \bar{u}' + \varepsilon^2 \bar{u}'' \quad (2.8)$$

$$n \doteq n^0 + \varepsilon n' + \varepsilon^2 n'' \quad (2.9)$$

Random domains and boundaries are incorporated into the formulation through transformation

$$d\Omega = J_V dV \quad (2.10)$$

$$d\Gamma = J_S dS \quad (2.11)$$

where V and S are the reference domain and boundary, respectively, J_V and J_S are the volume and surface Jacobians, respectively. J_V and J_S are expanded similarly up to the second order,

$$J_V \doteq J_V^0 + \varepsilon J_V' + \varepsilon^2 J_V'' \quad (2.12)$$

$$J_S \doteq J_S^0 + \varepsilon J_S' + \varepsilon^2 J_S'' \quad (2.13)$$

Substituting the expanded functions into eq. (2.1), and comparing the same order coefficient, the random variational principle is yielded, that is,

Zeroth-order variational principle

$$\int_V \delta e^T D^0 e^0 J_V^0 dV = \int_V \delta u^T \bar{f}^0 J_V^0 dV + \int_S \delta u^T \bar{p}^0 J_S^0 dS \quad (2.14)$$

First-order variational principle

$$\int_V \delta e^T (D^0 e' J_V^0 + D' e^0 J_V' + D^0 e^0 J_V') dV = \int_V \delta u^T (\bar{f}' J_V^0 + \bar{f}^0 J_V') dV$$

$$+ \int_{S_p} \delta u^T (\bar{p}' J_s^0 + \bar{p}^0 J_s') dS \quad (2.15)$$

Second-order variational principle

$$\begin{aligned} & \int_V \delta e^T (D^0 e'' J_v^0 + D' e' J_v^0 + D'' e^0 J_v^0 + D^0 e' J_v' + D^0 e^0 J_v' + D' e^0 J_v') dV \\ &= \int_V \delta u^T (\bar{f}'' J_v^0 + \bar{f}' J_v' + \bar{f}^0 J_v') dV + \int_{S_p} \delta u^T (\bar{p}^0 J_s^0 + \bar{p}' J_s' + \bar{p}'' J_s'') dS \end{aligned} \quad (2.16)$$

Since eq. (2.14) is the standard deterministic variational principle, the usual finite element method can be employed. The random functions D , \bar{f} , \bar{p} , J_v , J_s and the functions with the superscript (") in eqs. (2.15) and (2.16) are described through spatial expectation and autocovariance functions.

III. The Stochastic Finite Element Method

In the finite element analysis, the displacement field and the random field are discretized in the meantime.

The random field is discretized with q elements, That is

$$b(x) = \sum_{i=1}^q \phi_i(x) b_i \quad (3.1)$$

where $\phi_i(x)$ are shape functions. In element i , $\phi_i(x)=1$, otherwise $\phi_i(x)=0$, b_i are the central nodal values of $b(x)$ in i th element.

To maintain the convergence, the random functions D , \bar{f} , \bar{p} , \bar{u} , J_v and J_s are discretized with the same shape functions. For example, the approximation of random function D is

$$D = \sum_{i=1}^q \Phi_i(x) D_i = \sum_{i=1}^q \Phi_i(x) (D_i^0 + \varepsilon D_i' + \varepsilon^2 D_i'') \quad (3.2)$$

Expanding D_i as a Taylor series about \bar{b}

$$D_i = D_i^0 + \sum_{k=1}^q \left(\frac{\partial D_i}{\partial b_k} \right)_{\bar{b}} db_k + \frac{1}{2} \sum_{k=1}^q \sum_{l=1}^q \left(\frac{\partial^2 D_i}{\partial b_k \partial b_l} \right)_{\bar{b}} db_k db_l \quad (3.3)$$

so there exist

$$D_i' = \sum_{k=1}^q (D_i')_k \Delta b_k \quad (3.4)$$

$$D_i'' = \frac{1}{2} \sum_{k=1}^q \sum_{l=1}^q (D_i'')_{kl} \Delta b_k \Delta b_l \quad (3.5)$$

In eq. (3.3), $db_k = \varepsilon(b_k - \bar{b}_k) = \varepsilon \Delta b_k$, the nodal values $(D_i')_k$ and $(D_i'')_{kl}$ can be obtained by partial differentiation. For other random functions similar definition holds.

The displacement field is discretized with NL elements and NP nodes with each node having NF degrees of freedom. The approximation expression for the displacement field is

$$u = u^0 + \varepsilon u' + \varepsilon^2 u'' \quad (3.6)$$

thus

$$\mathbf{u}^0 = \sum_{i=1}^{NP} \mathbf{N}_i(\mathbf{x}) \mathbf{d}_i^0 = \mathbf{N}(\mathbf{x}) \mathbf{d}^0 \quad (3.7)$$

$$\mathbf{u}' = \sum_{i=1}^{NP} \mathbf{N}_i(\mathbf{x}) \mathbf{d}_i' = \mathbf{N}(\mathbf{x}) \mathbf{d}' \quad (3.8)$$

$$\mathbf{u}'' = \sum_{i=1}^{NP} \mathbf{N}_i(\mathbf{x}) \mathbf{d}_i'' = \mathbf{N}(\mathbf{x}) \mathbf{d}'' \quad (3.9)$$

where $\mathbf{N}(\mathbf{x})$ is the whole displacement shape function. \mathbf{d}^0 in eq. (3.7) are the whole nodal displacements, and $\mathbf{N}_i = \mathbf{N}_i^e$, when the point concerned is with a particular element e and i is a point associated with that element. If point i does not occur within the element $\mathbf{N}_i = \mathbf{0}$ where \mathbf{N}_i^e is element shape function \mathbf{N}_i^e which must be selected to satisfy the C_0 continuing requirement for displacement. \mathbf{d}_i' and \mathbf{d}_i'' are defined by

$$\mathbf{d}_i' = \sum_{k=1}^q (\mathbf{d}_i')_k \Delta b_k \quad (3.10)$$

$$\mathbf{d}_i'' = \frac{1}{2} \sum_{k=1}^q \sum_{l=1}^q (\mathbf{d}_i'')_{kl} \Delta b_k \Delta b_l \quad (3.11)$$

Substituting approximate expression into eqs. (2.14), (2.15) and (2.16) yields the stochastic finite element equations, that is,

Zeroth-order equations

$$\mathbf{K} \mathbf{d}^0 = \mathbf{F}^0 \quad (3.12)$$

where

$$\mathbf{K} = \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}^0 (\mathbf{L}\mathbf{N}) J_V^0 dV \quad (3.13)$$

$$\mathbf{F}^0 = \int_V \mathbf{N}^T \bar{\mathbf{f}}^0 J_V^0 dV + \int_{S_r} \mathbf{N}^T \bar{\mathbf{p}}^0 J_S^0 dS \quad (3.14)$$

\mathbf{d}^0 is the whole displacement vector. \mathbf{L} is strain operator.

First-order equation

$$\mathbf{K} \mathbf{d}_k' = \mathbf{F}_k' \quad (k=1, 2, \dots, q) \quad (3.15)$$

where

$$\begin{aligned} \mathbf{F}_k' = & \int_V \mathbf{N}^T [\bar{\mathbf{f}}_k' J_V^0 + \bar{\mathbf{f}}^0 (J_V')_k] dV + \int_{S_r} \mathbf{N}^T [\bar{\mathbf{p}}' J_S^0 + \bar{\mathbf{p}}^0 (J_S')_k] dS \\ & - \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_k' \mathbf{e}^0 J_V^0 dV - \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}^0 \mathbf{e}^0 (J_V')_k dV \end{aligned} \quad (3.16)$$

in eq. (3.16), $\mathbf{e}^0 = \mathbf{L}\mathbf{u}^0 = (\mathbf{L}\mathbf{N})\mathbf{d}^0$, $\bar{\mathbf{f}}_k' = (\partial \bar{\mathbf{f}} / \partial b_k)_{\bar{\mathbf{b}}}$ the other functions with superscripts (') have similar definition.

Second-order equations

$$\mathbf{K}\mathbf{d}'' = \mathbf{F}'' \quad (3.17)$$

where

$$\begin{aligned} \mathbf{F}'' = & \sum_{k=1}^q \sum_{i=1}^q \left\{ \frac{1}{2} \int_V \mathbf{N}^T \bar{\mathbf{f}}_{ki}^* J_{\bar{v}}^0 dV + \int_V \mathbf{N}^T \bar{\mathbf{f}}_{ki}^* (J_{\bar{v}}^1)_i dV + \frac{1}{2} \int_V \mathbf{N}^T \bar{\mathbf{f}}^0 (J_{\bar{v}}^2)_{ki} dV \right. \\ & + \frac{1}{2} \int_{S_r} \mathbf{N}^T [\bar{\mathbf{p}}_{ki}^* J_s^0 + 2\bar{\mathbf{p}}_k (J_s)_i + \bar{\mathbf{p}}^0 (J_s^2)_{ki}] dS - \frac{1}{2} \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_{ki}^* (\mathbf{L}\mathbf{N}) \mathbf{d}^0 J_{\bar{v}}^0 dV \\ & - \frac{1}{2} \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_{ki}^* (\mathbf{L}\mathbf{N}) \mathbf{d}^1 J_{\bar{v}}^1 dV - \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_{ki}^* (\mathbf{L}\mathbf{N}) \mathbf{d}^0 (J_{\bar{v}}^1)_i dV \\ & \left. - \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_0 (\mathbf{L}\mathbf{N}) \mathbf{d}^1 (J_{\bar{v}}^1)_i dV - \frac{1}{2} \int_V (\mathbf{L}\mathbf{N})^T \mathbf{D}_0 (\mathbf{L}\mathbf{N}) \mathbf{d}^0 (J_{\bar{v}}^2)_{ki} dV \right\} \Delta b_k \Delta b_i \quad (3.18) \end{aligned}$$

$$\mathbf{d}'' = \frac{1}{2} \sum_{k=1}^q \sum_{i=1}^q \mathbf{d}_{ki}^* \Delta b_k \Delta b_i \quad (3.19)$$

The mean value of displacements is

$$E[\mathbf{d}] = \mathbf{d}^0 + \bar{\mathbf{d}}'' \quad (3.20)$$

$$\bar{\mathbf{d}}'' = \frac{1}{2} \left\{ \sum_{k=1}^q \sum_{i=1}^q \mathbf{d}_{ki}^* \text{cov}(b_k, b_i) \right\} \quad (3.21)$$

The autocovariance matrices for displacements are

$$\text{cov}(d^i, d^j) = \left\{ \sum_{k=1}^q \sum_{i=1}^q \bar{\mathbf{d}}_{ki}^* \bar{\mathbf{d}}_{ki} \text{cov}(b_k, b_i) \right\} \quad (3.22)$$

in eq. (3.22), d^i is the i th degree of freedom of d , $\bar{\mathbf{d}}_{ki}^* = \left(\frac{\partial d^i}{\partial b_k} \right)_b$.

The mean value of stresses is

$$E(\boldsymbol{\sigma}) = \mathbf{D}^0 \mathbf{e}^0 + \left\{ \sum_{k=1}^q \sum_{i=1}^q \left[\mathbf{D}_{ki}^* \mathbf{B}(d^i)^{(m)} + \frac{1}{2} \mathbf{D}_{ki}^* \mathbf{B}(\mathbf{d}^0)^{(m)} \right] \text{cov}(b_k, b_i) \right\} \quad (3.23)$$

The autocovariance matrices are

$$\begin{aligned} \text{cov}(\boldsymbol{\sigma}^m, \boldsymbol{\sigma}^n) = & \left\{ \sum_{k=1}^q \sum_{i=1}^q [(\bar{\mathbf{D}}^m \mathbf{B}^m \bar{\mathbf{d}}_{ki}^*) (\bar{\mathbf{D}}^n \mathbf{B}^n \bar{\mathbf{d}}_{ki}^*)^T \right. \\ & + (\bar{\mathbf{D}}_{ki}^* \mathbf{B}^m \bar{\mathbf{d}}^m) (\bar{\mathbf{D}}_{ki}^* \mathbf{B}^n \bar{\mathbf{d}}^n)^T + \bar{\mathbf{D}}^m \mathbf{B}^m \bar{\mathbf{d}}_{ki}^* (\bar{\mathbf{D}}_{ki}^* \mathbf{B}^n \bar{\mathbf{d}}^n)^T \\ & \left. + (\bar{\mathbf{D}}_{ki}^* \mathbf{B}^m \bar{\mathbf{d}}^m) (\bar{\mathbf{D}}^n \mathbf{B}^n \bar{\mathbf{d}}_{ki}^*)^T] \text{cov}(b_k, b_i) \right\} \quad (3.24) \end{aligned}$$

where

$$\bar{\mathbf{D}}^m = \mathbf{D}^{0m}, \quad \bar{\mathbf{D}}_{ki}^* = \left(\frac{\partial D}{\partial b_k} \right)_b^m$$

IV. Numerical Example

A cantilever thin plate with random thickness subjected to a concentrated load F is shown in Fig.1 (a). Given the length of plate $l=40\text{cm}$, the height $H=10\text{cm}$, elasticity modulus $E=3.0 \times 10^9\text{N/cm}$, Poisson ratio $\nu=0.3$, $F=100\text{N}$, the mean value of the thickness $E[h]=1.0\text{cm}$, the standard deviation $\sigma=0.01$, the autocovariance function of the thickness is assumed to be

$$\text{cov}[h(x_r, y_r), h(x_s, y_s)] = \sigma^2 \exp\{[(x_r - x_s)/dx]^2\} \cdot \exp\{[(y_r - y_s)/dy]^2\}$$

where dx and dy are the decay factors in x and y direction, respectively. In this example $dx=40.00$, $dy=18.65$, statistics of displacements for P and Q are analysed.

The cantilever plate is discretized with 64 elements. The finite element modal is shown in Fig. 1 (b). The thickness of element is a random function, the displacement field is interpolated with linear functions. The random field is discretized similarly, i.e. with $q=64$. According to the

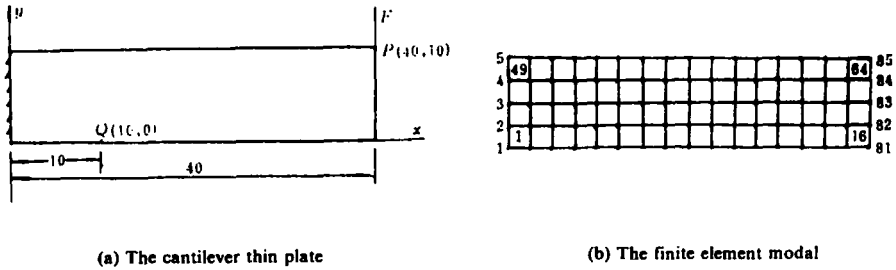


Fig. 1

programs developed by us, the computation results are shown in the following table.

The displacements of P and Q in y -direction

	$v(40, 10)$		$v(10, 0)$	
	mean value	std. deviation	mean value	std. deviation
This paper	0.00892	0.000838	0.000837	0.0000808
reference [6]	0.00879	0.000825	0.000830	0.0000801

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