

LARGE DEFLECTION ANALYSIS OF RECTANGULAR PLATES BY COMBINED PERTURBATION AND FINITE STRIP METHOD*

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Abstract

The perturbation method and finite strip method are combined to solve the large deflection bending problems of rectangular plates. Perturbation method is used to reduce the nonlinear differential equations into a series of linear differential equations. The finite strip method is then employed to tackle these linear equations. Some calculation examples are compared with those got by other methods.

Key words rectangular plate, large deflection, perturbation method, finite strip method

The finite strip method, with less input data and calculation compared with finite element method, is widely used in the solution of linear rectangular plate problems^[1]. But when it is extended to tackle the nonlinear problems, its advantages are partially obscured by the complexity of nonlinear numerical procedure^[2]. By first using the perturbation concept to reduce the nonlinear differential equations into a series of linear equations and then using the finite strip to solve these linear equations, we supply in the present paper another scheme for the solution of nonlinear rectangular plates with finite strip method, and so try to restore the neatness of finite strip method in nonlinear analysis somehow.

I. Perturbation of Nonlinear Plate Equations

As it is drawn in Fig. 1, a rectangular plate bends elastically under the distributed load q . Its displacements are u, v, w , respectively, along the x, y, z directions. If the plate material is elastically orthoptic, the nonlinear governing equations of the large deflection bending problems are,

$$U_{,\zeta\zeta} + \lambda^2 c_1 U_{,\eta\eta} + \lambda c_2 V_{,\zeta\eta} = -W_{,\zeta}(W_{,\zeta\zeta} + \lambda^2 c_1 W_{,\eta\eta}) - \lambda^2 c_2 W_{,\eta} W_{,\zeta\eta} \quad (1.1)$$

$$\lambda c_2 U_{,\zeta\eta} + c_1 V_{,\zeta\zeta} + \lambda^2 c_3 V_{,\eta\eta} = -\lambda W_{,\eta}(c_1 W_{,\zeta\zeta} + \lambda^2 c_3 W_{,\eta\eta}) - \lambda c_2 W_{,\zeta} W_{,\zeta\eta} \quad (1.2)$$

$$\begin{aligned} W_{,\zeta\zeta\zeta\zeta} + 2\lambda^2 c_4 W_{,\zeta\zeta\eta\eta} + \lambda^4 c_3 W_{,\eta\eta\eta\eta} &= P + W_{,\zeta\zeta}(U_{,\zeta} + \lambda v_{21} V_{,\eta}) \\ &+ \lambda^2 W_{,\eta\eta}(v_{21} U_{,\zeta} + \lambda c_3 V_{,\eta}) + 2\lambda c_1 W_{,\zeta\eta}(\lambda U_{,\eta} + V_{,\zeta} + \lambda W_{,\zeta} W_{,\eta}) \\ &+ \frac{1}{2} W_{,\zeta\zeta}(W_{,\zeta}^2 + \lambda^2 v_{21} W_{,\eta}^2) \\ &+ 2^{-1} \lambda^2 W_{,\eta\eta}(v_{21} W_{,\zeta}^2 + \lambda^2 c_3 W_{,\eta}^2) \end{aligned} \quad (1.3)$$

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where

$$\lambda = \frac{a}{b}, \quad \xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad c_1 = -\frac{\mu G_{12}}{E_1}, \quad c_2 = c_1 + \nu_{21}$$

$$c_3 = \frac{E_2}{E_1}, \quad c_4 = \nu_{21} + 2c_1, \quad \mu = 1 - \nu_{21}\nu_{12}, \quad U = \frac{12au}{h^2},$$

$$V = \frac{12av}{h^2}, \quad W = 2\sqrt{3}\frac{w}{h}, \quad P = 24\sqrt{3}\frac{qa^4\mu}{E_1h^4}$$

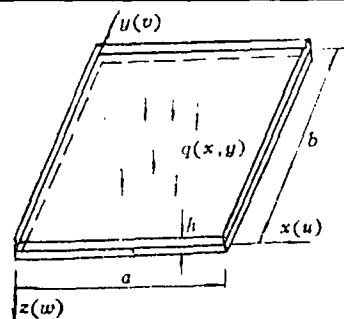


Fig. 1 Rectangular plate and load

The definitions of the elastic module E_1 , E_2 , etc. in the above formulas and the procedure of nondimensionalizations refer to reference [4].

As Qian and Yeh did^[3], we use the deflection of the central point of the plate as the perturbation parameter ϵ , i.e. we define,

$$\epsilon = W(1/2, 1/2) \quad (1.4)$$

Assume that the load could be put into the form $P = \varphi p^*(\xi, \eta)$, where p^* and φ specify the distribution and size of the load, respectively. Expand the displacement U , V , W and load parameter φ by ϵ in the following form,

$$W = \sum_{n=1}^{\infty} w^{(n)} \epsilon^n, \quad U = \sum_{n=1}^{\infty} u^{(n)} \epsilon^n, \quad V = \sum_{n=1}^{\infty} v^{(n)} \epsilon^n, \quad \varphi = \sum_{n=1}^{\infty} q^{(n)} \epsilon^n \quad (1.5 \sim 1.8)$$

By substituting (1.5) into the right-hand side of (1.4), and comparing the two sides of the resultant equation, the following conditions could be drawn.

$$w^{(1)}(1/2, 1/2) = 1; \quad w^{(n)}(1/2, 1/2) = 0 \quad (n \geq 2) \quad (1.9 \sim 1.10)$$

Applying expansions (1.5)–(1.8) in the governing equations (1.1)–(1.3) and perturbing them, i.e. establishing equalities by comparing the coefficients before ϵ^n on the two sides of the equations so got, we get a series of linear perturbation equations,

$$u_{,\xi\xi}^{(n)} + \lambda^2 c_1 u_{,\eta\eta}^{(n)} + c_2 \lambda v_{,\xi\eta}^{(n)} = -f_{n\xi}, \quad \lambda c_2 u_{,\xi\eta}^{(n)} + c_1 v_{,\xi\xi}^{(n)} + \lambda^2 c_3 v_{,\eta\eta}^{(n)} = -f_{n\eta} \quad (1.11 \sim 1.12)$$

$$w_{,\xi\xi\xi\xi}^{(n)} + 2\lambda^2 c_4 w_{,\xi\xi\eta\eta}^{(n)} + \lambda^4 c_3 w_{,\eta\eta\eta\eta}^{(n)} = -q^{(n)} p^*(\xi, \eta) + f_{nz} \quad (1.13)$$

where $f_{n\xi}$, $f_{n\eta}$ and f_{nz} are polynomials of $u^{(r)}$, $v^{(r)}$ and $w^{(r)}$ ($r \leq n-1$), and are zero when $n=1$. They are known and so could be treated as applied loads in each perturbation step. The formulas for the calculation of $f_{n\xi}$, $f_{n\eta}$ and f_{nz} are listed in the Appendix.

It is easy to find that equations (1.11) and (1.12) are just plane stress problems with the loads $f_{n\xi}$ and $f_{n\eta}$, and that equation (1.13) corresponds to linear bending of the plate under the load $-q^{(n)} p^*(\xi, \eta)$ and f_{nz} . The solutions of these equations will be proportional to parameter $q^{(n)}$, which could then be defined with the help of (1.9)–(1.10). So equations (1.9)–(1.13) are complete in that they give unique solutions for all the unknown terms in expansions (1.5)–(1.8).

Since equations (1.11)–(1.13) are linear partial differential equations, numerous methods exist for the solutions of them. But for plates of rectangular geometry which the present paper centers on the finite strip method is of special interest: it is simple in concept and has less amount of calculation than finite element method. In the next section, we will introduce the finite strip method into the calculation.

II. Perturbation Finite Strips

By the perturbation analysis of section I, the nonlinear plate bending problems are reduced to the solution of a series of linear plane stress and bending problems. For the solution of these linear problems with rectangular geometry by finite strip method, references are abundant. Here we adopt the displacement method^[1].

As is shown in Fig. 2, the plate is divided into s strips, with the displacement of the l 's strip for the n 's perturbation step assumed in the form,

$$u^{(n)} = \sum_{k=1}^m [u_k^{(n)}(1-\xi) + u_k^{(n)}\xi] X_k(\eta) \quad (2.1)$$

$$v^{(n)} = \sum_{k=1}^m [v_k^{(n)}(1-\xi) + v_k^{(n)}\xi] Y_k(\eta) \quad (2.2)$$

$$w^{(n)} = \sum_{k=1}^m [(1-3\xi^2+2\xi^3)w_k^{(n)} + \xi(1-2\xi+\xi^2)\theta_k^{(n)} + (3\xi^2 - 2\xi^3)w_k^{(n)} + \xi(\xi^2-\xi)\theta_k^{(n)}] Z_k(\eta) \quad (2.3)$$

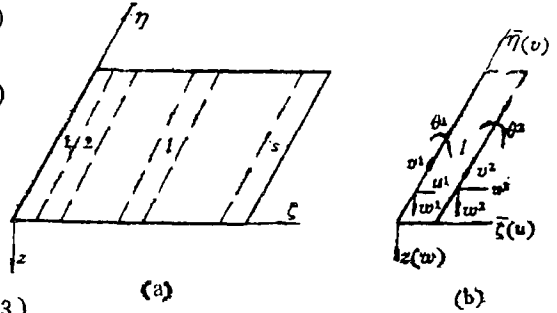


Fig. 2 Division of strip and its significance

where ξ is the local coordinate in the l 's strip, θ is the angle rotated on the sides of the strip, as denoted in Fig. 2 (b). The scripts, for example, of $u_k^{(n)}$ etc., mean that the quantity belongs to the n 's perturbation step, is on the i 's boundary of l 's strip, and corresponds to k 's term in the series expansion in the η direction.

The governing equations (1.11)–(1.13) in the matrix forms are,

$$K_p \Delta^{(n)} = G^{(n)}, \quad K_b \delta^{(n)} = q^{(n)} \bar{P} + F^{(n)} \quad (2.4 \sim 2.5)$$

$$\delta^{(n)} = [w_1^{(n)}, \theta_1^{(n)}, w_2^{(n)}, \theta_2^{(n)}, \dots, \theta_k^{(n)}, \dots]^T \quad (2.6)$$

$$\Delta^{(n)} = [u_1^{(n)}, v_1^{(n)}, u_2^{(n)}, v_2^{(n)}, \dots, v_k^{(n)}, \dots]^T \quad (2.7)$$

following some manipulations introduced in reference [1]. In the above formulas, K_p and K_b are the stiffness matrices of plane stress and bending problems, respectively. \bar{P} is the load column corresponding to p^* in equation (1.13), and $G^{(n)}$ and $F^{(n)}$ are the matrix forms of $f_{n\zeta}$. T is the transpose symbol.

We note that the formulation of $G^{(n)}$ and $F^{(n)}$ is rather complex. It can be observed from the formulas in the Appendix that $f_{n\zeta}$, $f_{n\eta}$ and $f_{n\zeta}$ are polynomials of $u^{(n)}$, $v^{(n)}$ and $w^{(n)}$ ($\gamma \leq n-1$) in two or three orders. Since $F^{(n)}$ and $G^{(n)}$ are got by integration of $f_{n\zeta}$, etc., after they are multiplied by virtual displacements, $G^{(n)}$ and $F^{(n)}$ are therefore polynomials of $u^{(n)}$, etc., in three or four orders. To simplify the finite strip realization of $F^{(n)}$ and $G^{(n)}$, the series expanded in the η direction are chosen to be trigonometrical ones matching special boundary conditions given, so integrations in this direction could be exactly carried out. As for the ξ direction, since displacements are assumed to be polynomials, the Gaussian integration formulas are used. In this way, we get results with rather high accuracy through relatively simple computer calculations.

III. Example Calculations

Based on the analysis of the above two sections, we compiled a Fortran program to realize the finite strip calculation in IBM PC computer. In this section, results of some example calculations are

reported. For the sake of easy comparison with results by other methods in literature, we assume that the loads are homogeneously distributed, i.e. $p^*(\xi, \eta) = 1$

Example 1 Clamped isotropic rectangular plate

For this case, $E_1 = E_2 = E$, $G_{12} = E/[2(1 + \nu)]$. $\nu_{12} = \nu_{21} = 1/3$. And the boundary conditions are,

$$u = v = w = w_{,xx} = 0; \quad \text{when } x = 0, a;$$

$$u = v = w = w_{,yy} = 0 \quad \text{when } y = 0, b.$$

The series in the η direction are therefore chosen to be,

$$X_n(\eta) = \sin n\pi\eta, \quad Y_n(\eta) = \sin n\pi\eta, \quad Z_n(\eta) = 1 - \cos 2n\pi\eta$$

The square plates ($\lambda = 1$) are calculated first. The final results are written as relations between the loading size parameter φ^* and the deflection in the center of the square, i.e. ϵ

$$\varphi^* = q^{(1)}\epsilon + q^{(3)}\epsilon^3 \quad (3.1)$$

where $\varphi^* = 2\sqrt{3}q \cdot (a/2)^4 / Dk$. The $q^{(1)}$ and $q^{(3)}$ calculated are listed in Table 1 where numbers of strips and terms of series in η direction included in each calculation are also specified.

Table 1 Results for clamped square plate

Number of strips	Terms of series			$q^{(1)}$	$q^{(3)}$
	u	v	w		
4	4	4	3	49.53	2.03
6	4	4	3	49.45	2.07
6	4	4	4	49.53	2.09
8	4	4	3	49.49	2.03
Results of ref. [3]				50.3815	2.03

The results by Qian and Yeh are also listed in the same table^[3]. Qian and Yeh got the results by perturbation method, too. But for the solution to the perturbation equation, they used polynomials of ξ and η with unknown parameters as trial function which were then fixed through some algebraic procedures. The results listed in Table 1 by the two methods are approximately the same. But we note that, when the boundary conditions or/and the load distributions are complex, Qian and Yeh's method will lead to rather lengthy calculation, and then the finite strip method will show its superiority.

For rectangular plates, $a \neq b$, cases of $\lambda = 1/3$ and $2/3$ as examples are calculated. In the calculations, we use 6 strips and include 4 terms of series in the η direction. The calculated results are shown in Fig. 3.

Example 2 Simply supported isotropic plate

The material of the plates is assumed to be the same as that in example 1. The boundary conditions are

$$u = v = w = w_{,xx} = 0 \quad \text{when } x = 0, a;$$

$$u = v = w = w_{,yy} = 0 \quad \text{when } y = 0, b.$$

The following form of series in the η direction is adopted,

$$X_n(\eta) = \sin n\pi\eta; \quad Y_n(\eta) = \sin n\pi\eta; \quad Z_n(\eta) = \sin n\pi\eta$$

The calculated results for the case $\lambda = 1$ are listed in Fig. 4.

From the above example calculations, we see that the combined method of perturbation and finite strip are rather effective for the solutions of large deflection rectangular problems.

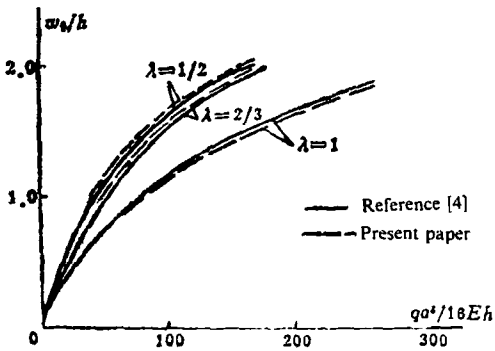


Fig. 3 Clamped plate

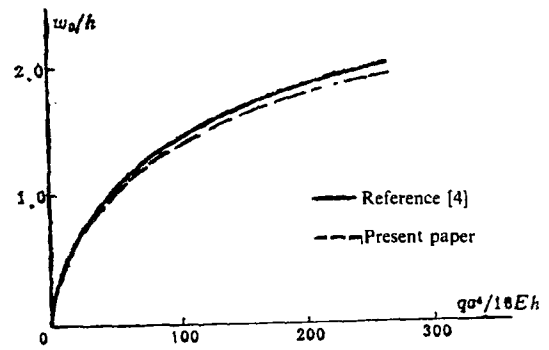


Fig. 4 Simply supported plate

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