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SINGULAR PERTURBATION OF INITIAL-BOUNDARY VALUE PROBLEMS FOR A CLASS OF REACTION DIFFUSION SYSTEMS*

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Abstract

In this paper, a class of singularly perturbed initial-boundary value problems for the reaction diffusion systems is considered. Using the theory of differential inequality, we prove that the initial-boundary value problems have a solution and obtain their asymptotic expansion.

Key words reaction diffusion system, singular perturbation, comparison theorem, asymptotic expansion

We consider the model of the reaction diffusion systems:

$$\partial u_i/\partial t-(\varepsilon L+L_1)u_i=f_i(t,x,u_1,u_2,\varepsilon) \quad (i=1,2)$$
$$(x\equiv(x_1,x_2,\cdots,x_n)\in\Omega,\ t\in(0,T]) \tag{1}$$
$$B_iu_i\equiv\alpha_i(x)\partial u_i/\partial\nu+\beta_i(x)u_i=g_i(x,\varepsilon) \quad (i=1,2)\quad x\in\partial\Omega \tag{2}$$
$$u_i(0,x,\varepsilon)=h_i(x,\varepsilon), \quad (i=1,2) \tag{3}$$

where ε is a small positive parameter, Ω denotes a bounded region in R^n, $\partial\Omega$ signifies a smooth boundary of Ω, $\partial/\partial\nu$ denotes the inner normal derivative on $\partial\Omega$, $\alpha_i(x)\leqslant\alpha_0<0$, $\beta_i(x)\geqslant\beta_0>0$, and L means a second order strong elliptic operator:

$$\left.\begin{aligned}&L\equiv\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}a_i(x)\frac{\partial}{\partial x_i}+a(x)\\&\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geqslant\delta_1>0, \qquad x\in\Omega\end{aligned}\right\} \tag{4}$$

where $\forall\,\xi_i\in R$ $(i=1,2,\ldots,n)$, $a_{ij}(x)=a_{ij}(x)$ and L_1 means a first order differential operator:

$$\left.\begin{aligned}&L_1\equiv\sum_{i=1}^{n}b_i(x)\frac{\partial}{\partial x_i}-b(x) \qquad b(x)\geqslant b_0>0\\&\left[\sum_{i=1}^{n}b_i(x)\nu_i\right]_{\partial\Omega}\geqslant\delta_2>0\end{aligned}\right\} \tag{5}$$

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Assume that $f_i, g_i, h_i, a_{ij}, a_i, a, b_i, b$. α_i and β_i are sufficiently smooth functions at the definited regions of their variables, α_0 , β_0 , b_0 , δ_1 and δ_2 are positive constants, ν_i is the direction coefficient of the inner normal on $\partial\Omega$ and there exists a positive constant l, which satisfies

$$f_{iu_1}+f_{iu_2}\leqslant -l<0 \qquad (i=1,\ 2) \tag{6}$$

The reaction diffusion equations are applied widely in the biophysics, biomathematics and physical chemistry etc. This problem is studied in many modern works, e.g. [1] – [5]. The problem (1) – (3) is a model of a class of problems which is considered widely. This paper makes use of the mothod[6] of multiple scales to solve the formal solution of the problem (1) – (3) and prove corresponding uniformly validity by using the differential inequality[7],[10]–[13].

Now we first construct the formal asymptotic solution of the problem (1) – (3). As $\varepsilon=0$, the reduced situation of the original problem becomes:

$$\begin{cases} \partial u_i/\partial t - L_1 u_i = f_i(t,x,u_1,u_2,0) & (7)\\ u_i(0,x,0)=h_i^!(x,0) & (8)\end{cases} \qquad (i=1,2)$$

where $h_i^!(x,0)$ is a sufficiently smooth function which is $h_i(x,0)$ to extend from Ω to R^n. (7) is a symmetric hyperbolic system. Assume that

$$u_i=U_{i0}(t,x) \qquad (i=1,\ 2) \tag{9}$$

is a group of sufficiently smooth solutions of Cauchy problem (7) and (8) in Ω. Let the outer solution of (1) – (3) be

$$U_i(t,x,\varepsilon)\sim\sum_{j=0}^{\infty}U_{ij}(t,x)\varepsilon^j \qquad (i=1,2) \tag{10}$$

Devoleping f_i and $h_i^!$ in ε , substituting (10) into (1), (3), equating coefficients of like powers of ε respectively and considering that U_{i0} satisfies (7) and (8), we obtain

$$\begin{cases} \partial U_{ij}/\partial t - L_1 U_{ij} = F_{ij} + LU_{i(j-1)} & (11)\\ U_{ij}(0,x)=h_{ij} & (12)\end{cases} \qquad (j=1,2,\cdots;\ i=1,2)$$

with

$$f_i(t,x,u_1,u_2,\varepsilon)\equiv F_i(\varepsilon)\sim\sum_{j=0}^{\infty}F_{ij}\varepsilon^j \qquad (i=1,2)$$

$$h_i^!(x,\varepsilon)\sim\sum_{j=0}^{\infty}h_{ij}\varepsilon^j \qquad (i=1,2)$$

$$\begin{aligned} F_{ij}&=\frac{1}{j!}\frac{d^jF_i(\varepsilon)}{d\varepsilon^j}\bigg|_{\varepsilon=0}\\ &=f_{iu_1}(t,x,U_{10},U_{20},0)U_{1j}+f_{iu_2}(t,x,U_{10},U_{20},0)U_{2j}\\ &\quad +C_{ij}(t,x,U_{10},U_{11},\cdots,U_{1(j-1)},U_{20},U_{21},\cdots,U_{2(j-1)})\\ &\qquad\qquad (j=0,1,2,\cdots)\end{aligned}$$

$$h_{ij}=\frac{1}{j!}\frac{\partial^jh_i^!}{\partial\varepsilon^j}\bigg|_{\varepsilon=0} \qquad (j=0,1,2,\cdots)$$

which C_{ij} is a determined function in j, whose construction is omitted.

From the linear problems (11) and (12), we can solve $U_{ij}(i=1,2)$ successively. Substituting them and (9) into (10), we obtain the outer solution for the original problem. But it may not satisfy boundary condition (2) so that we need to construct the boundary layer terms.

We now set up a local coordinate system (ρ, φ) , where $\varphi\equiv(\varphi_1,\varphi_2,\cdots,\varphi_{n-1})$ is a nonsingular coordinate system of the point on $(n-1)$-dimensional manifold $\partial\Omega$. Define the coordinate of every point Q in the neighborhood near $\partial\Omega$ in the following way: The coordinate $(\rho\leqslant\rho_0)$ is the distance from the point Q to the boundary $\partial\Omega$, where ρ_0 is small enough such that the inner normals on every point of $\partial\Omega$ do not intersect each other in this neighborhood near $\partial\Omega$. The coordinate φ of the point Q is equal to the coordinate φ of the point P at which the inner normal through the point Q intersects the boundary $\partial\Omega$.

In the above local coordinate system, in the neigborhood of $\partial\Omega$ $\rho\leqslant\rho_0'$ the original problem (1)−(3) is represented by

$$\partial u_i/\partial t-(\varepsilon\bar{L}+\bar{L}_1)u_i=\bar{f}_i(t,\rho,\varphi,u_1,u_2,\varepsilon)\qquad(i=1,2)\tag{13}$$

$$\bar{B}_i u_i\equiv[\bar{a}_i(\varphi)\partial u_i/\partial\rho+\bar{\beta}_i(\varphi)u_i]_{\partial\Omega}=\bar{g}_i(\varphi,\varepsilon)|_{\partial\Omega}\qquad(i=1,2)\tag{14}$$

$$u_i(0,\rho,\varphi,\varepsilon)=\bar{h}_i(\rho,\varphi,\varepsilon)\qquad(i=1,2)\tag{15}$$

where

$$\bar{L}\equiv\bar{a}_{nn}(\rho,\varphi)\frac{\partial^2}{\partial\rho^2}+\sum_{i=1}^{n-1}\bar{a}_{in}(\rho,\varphi)\frac{\partial^2}{\partial\rho\partial\varphi_i}+\sum_{i,j=1}^{n-1}\bar{a}_{ij}(\rho,\varphi)\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}$$

$$+\bar{a}_n(\rho,\varphi)\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{a}_i(\rho,\varphi)\frac{\partial}{\partial\varphi_i}+\bar{a}(\rho,\varphi)$$

$$\bar{L}_1\equiv\bar{b}_n(\rho,\varphi)\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{b}_i(\rho,\varphi)\frac{\partial}{\partial\varphi_i}-\bar{b}(\rho,\varphi)$$

$$\bar{f}_i(t,\rho,\varphi,u_1,u_2,\varepsilon)\equiv f_i(t,x,u_1,u_2,\varepsilon)$$

$$\bar{g}_i(\varphi,\varepsilon)\equiv g_i(x,\varepsilon)|_{\partial\Omega},\quad \bar{h}_i(\rho,\varphi,\varepsilon)\equiv h_i'(x,\varepsilon)$$

$$\bar{a}_i(\varphi)\equiv a_i(x)|_{\partial\Omega},\quad \bar{\beta}_i(\varphi)\equiv\beta_i(x)|_{\partial\Omega}$$

We lead in the variables of multiple scales[6]:

$$\tilde{\rho}\equiv k(\rho,\varphi)/\varepsilon,\quad \bar{\rho}\equiv\rho,\quad \varphi\equiv\varphi$$

where $k(\rho,\varphi)$ is an undetermind function decided below. Hence we have

$$\frac{\partial}{\partial\rho}=\frac{k_\rho}{\varepsilon}\frac{\partial}{\partial\tilde{\rho}}+\frac{\partial}{\partial\bar{\rho}}$$

$$\frac{\partial^2}{\partial\rho^2}=\frac{k_\rho^2}{\varepsilon^2}\frac{\partial^2}{\partial\tilde{\rho}^2}+2\frac{k_\rho}{\varepsilon}\frac{\partial^2}{\partial\tilde{\rho}\partial\bar{\rho}}+\frac{k_{\rho\rho}}{\varepsilon}\frac{\partial}{\partial\tilde{\rho}}+\frac{\partial^2}{\partial\bar{\rho}^2}$$

For convenience, we still substitute ρ for $\bar{\rho}$ below. Then

$$\partial/\partial t-(\varepsilon\bar{L}+\bar{L}_1)=\varepsilon^{-1}K_0+K_1+\varepsilon K_2$$

where

$$K_0\equiv-\left[\bar{a}_{nn}k_\rho^2\frac{\partial^2}{\partial\bar{\rho}^2}+\bar{b}_n k_\rho\frac{\partial}{\partial\bar{\rho}}\right] \tag{16}$$

$$K_1\equiv\frac{\partial}{\partial t}-\left[2\bar{a}_{nn}k_\rho\frac{\partial^2}{\partial\bar{\rho}\partial\rho}+\bar{a}_{nn}k_{\rho\rho}\frac{\partial}{\partial\bar{\rho}}+\sum_{i=1}^{n-1}\bar{a}_{in}k_\rho\frac{\partial^2}{\partial\rho\partial\varphi_i}+\bar{a}_n k_\rho\frac{\partial}{\partial\bar{\rho}}\right.$$

$$\left.+\bar{b}_n\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{b}_i\frac{\partial}{\partial\varphi_i}-\bar{b}\right] \tag{17}$$

$$K_2\equiv-\left[\bar{a}_{nn}\frac{\partial^2}{\partial\rho^2}+\sum_{i=1}^{n-1}\bar{a}_{in}\frac{\partial^2}{\partial\rho\partial\varphi_i}+\sum_{i,j=1}^{n-1}\bar{a}_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}+\bar{a}_n\frac{\partial}{\partial\rho}\right.$$

$$\left.+\sum_{i=1}^{n-1}\bar{a}_i\frac{\partial}{\partial\varphi_i}+\bar{a}\right] \tag{18}$$

We assume that a group of boundary layer terms $\bar{U}_i$ of the solution for the original problem(1)–(3) is

$$\bar{U}_i(t,\ \bar{\rho},\ \rho,\varphi,\varepsilon)\sim\sum_{j=1}^{\infty}\bar{U}_{ij}(t,\bar{\rho},\rho,\varphi)\varepsilon^j \qquad (i=1,2) \tag{19}$$

Let

$$u_i=U_i+\bar{U}_i \qquad (i=1,2) \tag{20}$$

Substituting (20) into (13), we have

$$\partial\bar{U}_i/\partial t-(\varepsilon\bar{L}+\bar{L}_1)\bar{U}_i=\bar{f}_i(t,\rho,\varphi,U_1+\bar{U}_1,U_2+\bar{U}_2,\varepsilon)-\bar{f}_i(t,\rho,\varphi,U_1,U_2,\varepsilon)$$
$$(i=1,2)$$

and substituting (16)–(19) into the above equatity, we obtain

$$\varepsilon^{-1}\sum_{j=1}^{\infty}(K_0[\bar{U}_{ij}]+K_1[\bar{U}_{i(j-1)}]+K_2[\bar{U}_{i(j-2)}])\varepsilon^j$$

$$=\bar{f}_i\left(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{1j}\varepsilon^j,\ \sum_{j=0}^{\infty}U_{2j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{2j}\varepsilon^j,\varepsilon\right)$$

$$-\bar{f}_i\left(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j,\sum_{j=0}^{\infty}U_{2j}\varepsilon^j,\varepsilon\right)\equiv\sum_{j=1}^{\infty}\bar{F}_{ij}\varepsilon^j \qquad (i=1,2) \tag{21}$$

The above and following values of $\bar{U}_{ij}$ and others for the nonpositive subscript are zero, while

$$\bar{F}_{ij}=\frac{1}{j!}\frac{\partial^j\bar{F}_i}{\partial\varepsilon^j}\bigg|_{\varepsilon=0} \qquad (j=1,2,\cdots;\ i=1,2)$$

$$\bar{F}_i\equiv\bar{f}_i\left(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{1j}\varepsilon^j,\ \sum_{j=0}^{\infty}U_{2j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{2j}\varepsilon^j,\varepsilon\right)$$

$$-f_i\left(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j,\sum_{j=0}^{\infty}U_{2j}\varepsilon^j,\varepsilon\right) \qquad (i=1,2)$$

Equating coefficients of like powers of ε for (21), we obtain

$$K_0[\bar{U}_{i1}]\equiv-\left[\bar{a}_{nn}k_\rho^2\frac{\partial^2\bar{U}_{i1}}{\partial\tilde{\rho}^2}+\bar{b}_n k_\rho\frac{\partial\bar{U}_{i1}}{\partial\tilde{\rho}}\right]=0 \qquad (i=1,2) \qquad (22)_1$$

$$K_0[\bar{U}_{ij}]=-K_1[\bar{U}_{i(j-1)}]-K_2[\bar{U}_{i(j-2)}]+\bar{F}_{i(j-1)} \qquad (j=2,3,\cdots;\ i=1,2) \qquad (22)_j$$

Let

$$k_\rho=\bar{b}_n(\rho,\varphi)/\bar{a}_{nn}(\rho,\varphi)$$

that is,

$$k(\rho,\varphi)=\int_0^\rho\frac{\bar{b}_n(s,\varphi)}{\bar{a}_{nn}(s,\varphi)}ds \qquad (23)$$

From $(22)_1$, we have

$$\bar{U}_{i1}=\beta_{i1}(t,\rho,\varphi)\exp[-\tilde{k}]=\beta_{i1}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon] \qquad (i=1,2) \quad (24)$$

where β_{i1} is an undetermined function decided below. From (4) and (5), it is easily seen that for sufficiently small $0<\rho\leqslant\rho_0\leqslant\rho_0'$, (23) decides $k(\rho,\varphi)>0$. The $\bar{U}_{i1}$ $(i=1,2)$ of (24) is a function possessing the boundary layer behavior.

Let the right-hand side of $(22)_j$ be equal to zero:

$$-K_1[\bar{U}_{i1}]+\bar{F}_{i1}=0 \qquad (25)$$

Substituting (24) into (25), we have

$$-\frac{\partial\beta_{i1}}{\partial t}+\Big[-2\bar{a}_{nn}k_\rho\frac{\partial\beta_{i1}}{\partial\rho}-\bar{a}_{nn}k_{\rho\rho}\beta_{i1}+\sum_{j=1}^{n-1}\bar{a}_{jn}k_\rho\frac{\partial^2\beta_{i1}}{\partial\rho\partial\varphi_j}-\bar{a}_n k_\rho\beta_{i1}$$

$$+\bar{b}_n\frac{\partial\beta_{i1}}{\partial\rho}+\sum_{j=1}^{n-1}\bar{b}_j\frac{\partial\beta_{i1}}{\partial\varphi_j}-\bar{b}\beta_{i1}\Big]$$

$$=-[\bar{f}_{iu_1}(t,\rho,\varphi,U_{10},U_{20},0)\beta_{11}+\bar{f}_{iu_2}(t,\rho,\varphi,U_{10},U_{20},0)\beta_{21}] \qquad (i=1,2) \qquad (26)$$

Substituting (20) into (14), (15) and equating the coefficients of like powers of ε, we obtain

$$\left[\tilde{a}_i k_\rho\frac{\partial\bar{U}_{i1}}{\partial\tilde{\rho}}\right]_{\partial\Omega}=[g_{i0}-B_iU_{i0}]_{\partial\Omega} \qquad (i=1,2) \qquad (27)_1$$

$$\left[\tilde{a}_i k_\rho\frac{\partial\bar{U}_{ij}}{\partial\tilde{\rho}}\right]_{\partial\Omega}=\left[g_{i(j-1)}-B_iU_{i(j-1)}-\tilde{a}_i\frac{\partial\bar{U}_{i(j-1)}}{\partial\rho}\bar{\beta}_i-\bar{U}_{i(j-1)}\right]_{\partial\Omega}$$

$$(j=2,3,\cdots;\ i=1,2) \qquad (27)_j$$

$$\bar{U}_{ij}|_{t=0}=0 \qquad (j=1,2,\cdots;\ i=1,2) \qquad (28)_j$$

where

$$g_{ij}=\frac{1}{j!}\frac{\partial^j g_i}{\partial\varepsilon^j}\bigg|_{\varepsilon=0} \qquad (j=0,1,2,\cdots;\ i=1,2)$$

Substituting (24) into $(27)_1$ and $(28)_1$, we admit

$$\beta_{i1}|_{\partial\Omega}=\left[-\frac{1}{\bar{a}_i k_\rho}(\bar{g}_{i0}-B_iU_{i0})\right]_{\partial\Omega}\qquad(i=1,2)\tag{29$_1$}$$

$$\beta_{i1}(0,\rho,\varphi)=0\tag{30}$$

From the linear problems (26), $(29)_1$ and $(30)_1$ we derive $\beta_{i1}(t,\rho,\varphi)$, which is substituted into (24), and then we obtain $\bar{U}_{i1}(i=1,2)$. Considering $(22)_j$ and noting (25) we find a group of solutions:

$$\bar{U}_{i2}=\beta_{i2}(t,\rho,\varphi)\exp[-\bar{\rho}]=\beta_{i2}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon]$$
$$(i=1,2)\tag{31}$$

where β_{i2} is defind by the system of linear partial differential equations which equates the right-hand side of $(22)_3$ to zero and by the boundary and initial conditions $(27)_2$ and $(28)_2$. Using this way, we can define successively

$$\bar{U}_{ij}=\beta_{ij}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon]\qquad(j=3,4,\cdots;\ i=1,2)\tag{32}$$

Clearly, $\bar{U}_{ij}$ is a function possessing boundary layer behavior.

Let

$$\tilde{U}_{ij}=\psi\bar{U}_{ij}\qquad(j=1,2,\cdots;\ i=1,2)\tag{33}$$

where ψ is a sufficiently smooth function on $\Omega+\partial\Omega$ and satisfies

$$\psi=\begin{cases}1, & 0\leqslant\rho\leqslant\rho_0/3\\ 0, & 2\rho_0/3\leqslant\rho \text{ and other points but except } \rho\leqslant\rho_0.\end{cases}$$

Then we can construct the following formal asymptotic solution of original problem (1) – (3):

$$u_i\sim\sum_{j=0}^{\infty}U_{ij}\varepsilon^j+\sum_{j=1}^{\infty}\tilde{U}_{ij}\varepsilon^j\tag{34}$$

Now we proceed to prove the uniformly valid expansions of (34).

We first construct the functions $m_i(t,x,\varepsilon)$ and $M_i(t,x,\varepsilon)$ $(i=1,2)$:

$$m_i(t,x,\varepsilon)=Z_{im}(t,x,\varepsilon)-(\gamma/l)\varepsilon^m\tag{35}$$

$$M_i(t,x,\varepsilon)=Z_{im}(t,x,\varepsilon)+(\gamma/l)\varepsilon^m\tag{36}$$

where

$$Z_{im}(t,x,\varepsilon)=\sum_{j=0}^{m-1}U_{ij}\varepsilon^j+\sum_{j=1}^{m}\tilde{U}_{ij}\varepsilon^j\qquad(i=1,2)\tag{37}$$

and γ is a large enough undetermined positive constant.

Obviously, from (35), (36) and (37), it is easily seen that m_i and M_i are sufficiently smooth functions for variables t,x and satisfy

$$m_i(t,x,\varepsilon)<M_i(t,x,\varepsilon)\qquad(i=1,2)\tag{38}$$

From

$$\bar{g}_i(\varphi,\varepsilon)=\sum_{j=0}^{m-1}\bar{g}_{ij}\varepsilon^j+O(\varepsilon^m)\qquad(i=1,2)\quad(0<\varepsilon\ll1)$$

then selecting ε' small enough as $0<\varepsilon\leqslant\varepsilon'$, there is a positive d', such that

$$\left|g_i(\varphi,\varepsilon)-\sum_{j=0}^{m-1}g_{ij}\varepsilon^j\right|\leqslant d'\varepsilon^m \qquad (i=1,2)$$

Therefore

$$B[m_i(t,x,\varepsilon)]|_{\partial\Omega}=B[Z_{im}(t,x,\varepsilon)]|_{\partial\Omega}-\beta_i(x)\frac{\gamma}{l}\varepsilon^m\Big|_{\partial\Omega}$$

$$\leqslant g_i(\varphi,\varepsilon)|_{\partial\Omega}+\left(d'-\frac{\beta_0\gamma}{l}\right)\varepsilon^m \qquad (i=1,2)$$

Selecting γ large enough as $\gamma\geqslant\gamma'=ld'/\beta_0$, we have

$$B[m_i(t,x,\varepsilon)]|_{\partial\Omega}\leqslant g_i(\varphi,\varepsilon)|_{\partial\Omega}\doteq g_i(x,\varepsilon)|_{\partial\Omega} \qquad (i=1,2) \tag{39}$$

Analogously, we have

$$B[M_i(t,x,\varepsilon)]|_{\partial\Omega}\geqslant g_i(x,\varepsilon)|_{\partial\Omega} \qquad (i=1,2) \tag{40}$$

We now prove some differential inequalities below.

(i) As $x\in\Omega$ but except $\rho\leqslant 2\rho_0/3$, from (33), $\bar{U}_{ij}=0$ ($j=1,2,\ldots$; $i=1,2,$), then

$$m_i(t,x,\varepsilon)=\sum_{j=0}^{m-1}U_{ij}\varepsilon^j-\frac{\gamma}{l}\varepsilon^m \qquad (i=1,2)$$

$$\frac{\partial m_i}{\partial t}-(\varepsilon L+L_1)m_i=\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j+(\varepsilon a(x)-b(x))\frac{\gamma}{l}\varepsilon^m$$

$$\leqslant\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1} \qquad (i=1,2)$$

and bec

$$f_i\left(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\right)=\sum_{i=0}^{m-1}F_{ij}\varepsilon^j+O(\varepsilon^m)$$

$$(i=1,2)\quad 0<\varepsilon\ll 1$$

therefore for $\varepsilon_1>0$ small enough as $0<\varepsilon\leqslant\varepsilon_1$, there are $d_1>0$, $M>0$, such that

$$\left|f_i\left(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\ \sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\right)-\sum_{j=0}^{m-1}F_{ij}\varepsilon^j\right|\leqslant d_1\varepsilon^m \qquad (i=1,2)$$

$$|L[U_{i(m-1)}]|\leqslant M \qquad (i=1,2)$$

$$|a(x)|\leqslant M$$

From (6) and the mean value theorem, there are $0<\theta_1,\theta_2<1$, such that

$$f_i(t,x,m_1,m_2,\varepsilon)=f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)+\Big[\frac{\partial}{\partial u_1}f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j$$

$$+\theta_1\Big(m_1-\sum_{j=0}^{m-1}U_{1j}\varepsilon^j\Big)m_2,\varepsilon\Big)\Big]\Big(m_1-\sum_{j=0}^{m-1}U_{1j}\varepsilon^j\Big)+\Big[\frac{\partial}{\partial u_2}f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j$$

$$+\theta_2\Big(m_2-\sum_{j=0}^{m-1}U_{2j}\varepsilon^j\Big),\ \varepsilon\Big)\Big]\Big(m_2-\sum_{j=0}^{m-1}U_{2j}\varepsilon^j\Big)$$

$$\geqslant f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\ \sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)+\gamma\varepsilon^m\qquad(i=1,2)$$

then as $0<\varepsilon\leqslant\varepsilon_1'=\min(\varepsilon_1,l/2M)$,

$$\partial m_i/\partial t-(\varepsilon L+L_1)m_i-f_i(t,x,m_1,m_2,\varepsilon)$$

$$\leqslant\sum_{j=0}^{m-1}\Big[\frac{\partial U_{ij}}{\partial t}(\varepsilon L+L_1)U_{ij}\Big]\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1}-f_i(t,x,m_1,m_2,\varepsilon)$$

$$\leqslant\Big[\frac{\partial U_{i0}}{\partial t}-L_1[U_{i0}]-f_i(t,x,U_{10},U_{20},0)\Big]$$

$$+\sum_{j=1}^{m-1}\Big(\frac{\partial U_{ij}}{\partial t}-L_1[U_{ij}]-L[U_{i(j-1)}]-F_{ij}\Big)\varepsilon^j$$

$$-L[U_{i(m-1)}]\varepsilon^m+(d_1-\gamma/2)\varepsilon^m$$

$$\leqslant(M+d_1-\gamma/2)\varepsilon^m\qquad(i=1,2)$$

(ii) As $\rho_0/3\leqslant\rho\leqslant2\rho_0/3$, because $\bar{U}_{ij}$ $(j=1,2,\ldots,m;\ i=1,2)$ and all its partial derivatives asymptotically tend to zero and they are $o(\varepsilon^m)$, so we can obtain estimation using method of (i) for $\varepsilon_2'>0$ small enough as $0<\varepsilon\leqslant\varepsilon_2'$.

(iii) As $0<\rho\leqslant\rho_0/3$, $\tilde{U}_{ij}=\bar{U}_{ij}$ $(j=1,2,\ldots,m;\ i=1,2)$, then

$$m_i(t,x,\varepsilon)=\sum_{j=0}^{m-1}U_{ij}\varepsilon^j+\sum_{j=1}^{m}\bar{U}_{ij}\varepsilon^j-\frac{\gamma}{l}\varepsilon^m$$

$$\frac{\partial m_i}{\partial t}-(\varepsilon L+L_1)m_i=\sum_{j=0}^{m-1}\Big[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\Big]\varepsilon^j$$

$$+\sum_{j=1}^{m}\Big[\frac{\partial\bar{U}_{ij}}{\partial t}-(\varepsilon L+L_1)\bar{U}_{ij}\Big]\varepsilon^j+(\varepsilon a(x)-b(x))\frac{\gamma}{l}\varepsilon^m$$

$$\leqslant\sum_{j=0}^{m-1}\Big[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\Big]\varepsilon^j+\varepsilon^{-1}\sum_{j=1}^{m}(K_0[\bar{U}_{ij}]$$

$$+K_1[\bar{U}_{i(j-1)}]+K_2[\bar{U}_{i(j-2)}])\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1}$$

$$-(K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}])\varepsilon^m$$

and from

$$f_i(t,x,Z_{1m},Z_{2m},\varepsilon)=\sum_{j=0}^{m-1} F_{ij}\varepsilon^j+\sum_{j=0}^{m-1}\bar{F}_{ij}\varepsilon^j+O(\varepsilon^m)$$

$$(i=1,2) \qquad 0<\varepsilon\ll 1$$

for $\varepsilon_3>0$ small enough as $0<\varepsilon\leqslant\varepsilon_3$, there are $d_3>0, M'>0$, such that

$$\left|f_i(t,x,Z_{1m},Z_{2m},\varepsilon)-\sum_{j=0}^{m-1}F_{ij}\varepsilon^j-\sum_{j=0}^{m-1}\bar{F}_{ij}\varepsilon^j\right|\leqslant d_3\varepsilon^m \qquad (i=1,2)$$

$$|L[U_{i(m-1)}]|\leqslant M' \qquad (i=1,2)$$
$$|K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}]|\leqslant M' \qquad (i=1,2)$$
$$|a(x)|\leqslant M'$$

From the mean value theorem, we have

$$f_i(t,x,m_1,m_2,\varepsilon)\geqslant f_i(t,x,Z_{1m},Z_{2m},\varepsilon)+\gamma\varepsilon^m \qquad (i=1,2)$$

then as $0<\varepsilon\leqslant\varepsilon_3'=\min(\varepsilon_3,\ l/2M')$,

$$\partial m_i/\partial t-(\varepsilon L+L_1)m_i-f_i(t,x,m_1,m_2,\varepsilon)$$

$$\leqslant\left[\frac{\partial U_{i0}}{\partial t}-L_1[U_{i0}]-f_i(t,x,U_{10},U_{20},0)\right]+\sum_{j=1}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-L_1[U_{ij}]\right.$$

$$\left.-L[U_{i(j-1)}]-F_{ij}\right]\varepsilon^j+\varepsilon^{-1}\sum_{j=1}^{m}(K_0[\bar{U}_{ij}]+K_1[\bar{U}_{i(j-1)}]$$

$$+K_2[\bar{U}_{i(j-2)}]-\bar{F}_{i(j-1)})\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^m-L[U_{i(m-1)}]\varepsilon^m$$

$$-(K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}])\varepsilon^m+(d_3-\gamma)\varepsilon^m$$
$$\leqslant(2M'+d_3-\gamma/2)\varepsilon^m \qquad (i=1,2)$$

To sum up (i)–(iii), selecting $\varepsilon_0=\min(\varepsilon_1',\varepsilon_2',\varepsilon_3')$ and γ_0 large enough as $0<\varepsilon\leqslant\varepsilon_0,\gamma\geqslant\gamma_0$, we yield differential inequality

$$\partial m_i/\partial t-(\varepsilon L+L_1)m_i\leqslant f_i(t,x,m_1,m_2,\varepsilon) \qquad (i=1,2)$$
$$0<\varepsilon\leqslant\varepsilon_0,\ (t,x)\in(0,T]\times\Omega \qquad (41)$$

Analogously, we have

$$\partial M_i/\partial t-(\varepsilon L+L_1)M_i\geqslant f_i(t,x,M_1,M_2,\varepsilon) \qquad (i=1,2)$$
$$0<\varepsilon\leqslant\varepsilon_0,\ (t,x)\in(0,T]\times\Omega \qquad (42)$$

Finally, from (9) and $(28)_j$, it is easily seen that

$$m_i(0,x,\varepsilon)\leqslant h_i(x,\varepsilon)\leqslant M_i(0,x,\varepsilon) \qquad (i=1,2) \quad (0<\varepsilon\ll 1) \qquad (43)$$

From (38)–(43), using the comparison theorem[8],[9], for $\varepsilon>0$ small enough, there is a group of solutions $u_i(t,x,\varepsilon)$ $(i=1,2)$ for the initial boundary value problem(1)–(3), which satisfies the inequality

$$m_i(t,x,\varepsilon)\leqslant u_i(t,x,\varepsilon)\leqslant M_i(t,x,\varepsilon) \qquad (i=1,2) \quad (t,x)\in[0,T]\times(\Omega+\partial\Omega)$$

Then we obtain uniformly valid expansion

$$u_i(t,x,\varepsilon)=\sum_{j=0}^{m-1} U_{ij}\varepsilon^j+\sum_{j=1}^{m}\bar{U}_{ij}\varepsilon^j+O(\varepsilon^m) \qquad (i=1,2) \quad (0<\varepsilon\ll 1)$$

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