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REMARKS OF SOME PROBLEMS FOR RECTANGULAR THIN PLATES WITH FREE EDGES ON ELASTIC FOUNDATIONS

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Abstract

For the bending, stability and vibrations of rectangular thin plates with free edges on elastic foundations, in this paper we give a flexural function which exactly satisfies not only all the boundary conditions on free edges but also the conditions at free corner points. Applying energy variation principle, we give equations defining parameters in flexural function, stability equation, frequency equation, and general formulae of minimum critical force and minimum eigenfrequency as well.

Key words bending, flexural function, stability, vibration, critical force, frequency

I. Introduction

V.Z. Vlasov[1], Y.S. Kononenko[2] and Zhang Fo-van[3] researched the bending, stability and vibrations of rectangular thin plates with free edges on elastic foundations, but their methods are complicated. Paper [4] discussed the above problems, but we point out that the flexural functions does not satisfy conditions that moment of bending must be zero on boundary (see the end of the paper), and the main results were only for a case of square plate. This paper gives a flexural function which exactly satisfies all the boundary conditions on free edges and at corner points. For general rectangular plate, we give general formulae of problems and prove total antiforces on foundations equal to total loads on the plate. Hence the method is not only simple but also reliable. Finally, we give results for square plate $a=b$, $\mu=0.167$.

II. The Problem and Flexural Function

Supposing a rectangular thin plate on elastic fundations, the middle plane is $\Omega=\{x, y\,|\,0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b\}$ and the boundaries of the plate are free that is, boundary conditions are: on the $x=0$, a,

$$w_{xx}+\mu w_{yy}=0 \tag{2.1}$$

$$w_{xxx}+(2-\mu)w_{xyy}=0 \tag{2.2}$$

on the $y=0$, b,

$$w_{yy}+\mu w_{xx}=0 \tag{2.3}$$

$$w_{yyy}+(2-\mu)w_{yxx}=0 \tag{2.4}$$

at four corner points

$$w_{xy}=0 \tag{2.5}$$

We give the following flexural function

$$w=f_1\cos\frac{2\pi}{a}x+f_2\cos\frac{2\pi}{b}y+f_3\cos\frac{2\pi}{a}x\cos\frac{2\pi}{b}y$$
$$+f_4x(x-a)+f_5y(y-b)+f_6 \tag{2.6}$$

where a, b are respectively length of thin plate along x-axis and y-axis, coefficients $f_1\sim f_6$ will be defined parameters. Without difficulty we prove that function (2.6) satisfies conditions (2.2), (2.4) and (2.5), and substituting (2.6) into expression (2.1) and (2.3), we obtain:

$$\left.\begin{array}{ll} f_1=-\left(\mu+\frac{a^2}{b^2}\right)\frac{f_3}{\mu}, & f_2=-\left(\mu+\frac{b^2}{a^2}\right)\frac{f_3}{\mu} \\ f_4=-\frac{2\pi^2}{\mu b^2}f_3, & f_5=-\frac{2\pi^2}{\mu a^2}f_3 \end{array}\right\} \tag{2.7}$$

or

$$f_1=\beta_1 f_3,\quad f_2=\beta_2 f_3,\quad f_4=\beta_4\frac{\pi^2}{b^2}f_3,\quad f_5=\beta_5\frac{\pi^2}{a^2}f_3 \tag{2.8}$$

Such the flexural function (2.6) may be written

$$w=f_3\left[\beta_1\cos\frac{2\pi}{a}x+\beta_2\cos\frac{2\pi}{b}y+\cos\frac{2\pi}{a}x\cos\frac{2\pi}{b}y\right.$$
$$\left.+\beta_4\frac{\pi^2}{b^2}x(x-a)+\beta_5\frac{\pi^2}{a^2}y(y-b)\right]+f_6 \tag{2.9}$$

and so flexural function (2.9) satisfies all conditions (2.1)–(2.5), where it includes only two parameters f_3, f_6 will be defined.

III. The Problem of Bending

If the plate is acted on by outer loads, deformation potential energy is

$$U=\frac{D}{2}\iint[w_{xx}^2+w_{yy}^2+2\mu w_{xx}w_{yy}+2(1-\mu)w_{xy}^2]dxdy+\frac{K}{2}\iint w^2dxdy \tag{3.1}$$

where D is bending rigidity of the plate, K is coefficient of elastic foundatons, the domain of double integral is Ω, and the potential energy of outer forces is

$$W=\iint pw(x,y)dxdy \tag{3.2}$$

for the case of uniform distribution loads

$$W=p\iint wdxdy=-\frac{pab\pi^2}{6}\left(\beta_4\lambda^2+\frac{\beta_5}{\lambda^2}\right)f_3+pabf_6 \tag{3.3}$$

for the case of concentrated force at the center of the plate

$$W=Pw\left(\frac{a}{2},\ \frac{b}{2}\right)=P\left(-\beta_1-\beta_2+1-\frac{\beta_4\pi^2}{4}\lambda^2-\frac{\beta_5\pi^2}{4\lambda^2}\right)f_3+Pf_6 \tag{3.4}$$

where $\lambda = a/b$. Based on energy variation principle, when the plate is in stable equilibrium state, total potential energy $\Pi = U - W$ must be minimum, namely, $\delta\Pi = 0$. Hence variating two independent parameters, we obtain algebraic equations including f_6.

$$\left.\begin{aligned} &Af_3 - \frac{Kab\pi^2}{6}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)f_6 = B \\ &-\frac{Kab\pi^2}{6}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)f_3 + Kabf_6 = C \end{aligned}\right\} \tag{3.5}$$

where

$$B = \begin{cases} -\dfrac{ab\pi^2}{6}\left(\beta_4\lambda^2 + \dfrac{\beta_5}{\lambda^2}\right)p \text{ , the case of distribution loads,} \\ \left(-\beta_1 - \beta_2 + 1 - \dfrac{\beta_4\pi^2}{4}\lambda^2 - \dfrac{\beta_5\pi^2}{4\lambda^2}\right)P\text{, the case of concentrated load,} \end{cases} \tag{3.6}$$

$$C = \begin{cases} pab, & \text{the case of distribution loads,} \\ P, & \text{the case of concentrated load,} \end{cases} \tag{3.7}$$

$$\begin{aligned} A = &\frac{2D\pi^4}{a^2\lambda}\left[4(\beta_1^2 + \lambda^4\beta_2^2) + 2(1+\lambda^2)^2 + 2\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right)\right. \\ &\left. + \mu\beta_4\beta_5\lambda^2\right] + K\left[\frac{1}{2}\left(\beta_1^2 + \beta_2^2 + \frac{1}{2}\right) + \frac{\pi^4}{30}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right)\right. \\ &\left. + \beta_1\beta_4\lambda^2 + \frac{\beta_2\beta_5}{\lambda^2} + \frac{\beta_4\beta_5\pi^4}{18}\right]ab \end{aligned} \tag{3.8}$$

And so by expression (3.5) we may find f_3, f_6, and furthermore, we may find deflection and inner forces.

To pay attention to the second equation of expression (3.5), the total antiforces of foundations

$$R_f = K\iint w\,dx\,dy = -\frac{Kab\pi^2}{6}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)f_3 + Kabf_6 = \begin{cases} pab, \\ P. \end{cases} \tag{3.9}$$

Hence for general rectangular thin plate, the total antiforces of foundatons with outer total loads is always in equilibrium.

IV. Problems of Stability and Vibration

(1) We know distribution force p_x and p_y along directions of x-axis and y-axis act respectively on the free edges of the plate on elastic foundations, and then the potential energy of outer forces is

$$W = \frac{1}{2}\iint (p_x w_x^2 + p_y w_y^2)\,dx\,dy \tag{4.1}$$

Supposing $p_y = \gamma^0 p_x$, γ^0 is the the parameter of critical force, and then we have

$$\begin{aligned} W = &\frac{p_x f_3^2}{2}\left\{\frac{2\pi^2}{\lambda}\left(\beta_1^2 + \frac{1}{2}\right) + \lambda\beta_4\left(\frac{1}{3}\lambda^2\beta_4 + 4\beta_1\right)\right. \\ &\left. + \gamma^0\left[\frac{2\pi^2}{\lambda}\left(\beta_2^2 + \frac{1}{2}\right) + \lambda\beta_5\left(\frac{1}{3}\lambda^2\beta_5 + 4\beta_2\right)\right]\right\} = \frac{1}{2}p_x I f_3^2 \end{aligned} \tag{4.2}$$

Similarly, we obtain algebraic equations including parameters f_3, f_6

$$\left.\begin{aligned} &A_1 f_3 - \frac{Kab\pi^2}{6}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right) f_6 = 0 \\ &-\frac{Kab\pi^2}{6}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right) f_3 + Kabf_6 = 0 \end{aligned}\right\} \tag{4.3}$$

where $A_1 = A - p_x I$. In order that equations (4.3) may have nonzero solution its coefficient determinant must equal zero. Therefore we obtain the stability equation

$$A - p_x I - \frac{Kab\pi^4}{36}(\beta_4\lambda^2 + \beta_5/\lambda^2)^2 = 0 \tag{4.4}$$

from expression (4.4), we find minimum critical load as follows.

$$\begin{aligned} (p_{cr})_{\min} &= \frac{1}{I}\left[A - \frac{Kab\pi^4}{36}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)^2\right] \\ &= \frac{1}{I}\left\{\frac{2D\pi^4}{a^2\lambda}\left[4(\beta_1^2 + \lambda^4\beta_2^2) + 2(1+\lambda^2)^2\right.\right. \\ &\quad \left. + 2(\beta_4^2\lambda^4 + \beta_5^2) + \mu\beta_4\beta_5\lambda^2\right] + K\left[\frac{1}{2}\left(\beta_1^2 + \beta_2^2 + \frac{1}{2}\right)\right. \\ &\quad \left.\left. + \frac{\pi^4}{180}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right) + \beta_1\beta_4\lambda^2 + \frac{\beta_2\beta_5}{\lambda^2}\right]\right\} > 0 \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} I = \pi^2 &\left\{\frac{2}{\lambda}\left(\beta_1^2 + \frac{1}{2}\right) + \lambda\beta_4\left(\frac{1}{3}\lambda^2\pi^2\beta_4 + 4\beta_1\right)\right. \\ &\left. + \gamma^0\left[\frac{2}{\lambda}\left(\beta_2^2 + \frac{1}{2}\right) + \lambda\beta_5\left(\frac{1}{3}\lambda^2\pi^2\beta_5 + 4\beta_2\right)\right]\right\} \end{aligned} \tag{4.6}$$

(2) We know the moving energy of the thin plate with distribution masses is

$$W_k = \frac{1}{2}\omega^2\frac{\gamma h}{g}\iint w^2 dxdy \tag{4.7}$$

where ω is eigenfrequency of thin plate free vibrations, γ and g are respectively specific gravity and gravitation acceleration of the plate material. Substituting expression (2.9) into the above expression, we have

$$\begin{aligned} W_k = \frac{ab\omega^2\gamma h}{2g}&\left\{\left[\frac{\beta_1^2}{2} + \frac{\beta_2^2}{2} + \frac{1}{4} + \frac{\pi^4}{30}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right) + \beta_1\beta_4\lambda^2\right.\right. \\ &\left.\left. + \frac{\beta_2\beta_5}{\lambda^2} + \frac{\pi^4}{18}\beta_4\beta_5\right] f_3^2 + f_6^2 - \frac{\pi^2}{3}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right) f_3 f_6\right\} \end{aligned} \tag{4.8}$$

Substituting W by W_k and take variation, we have

$$\left.\begin{aligned} &A_2 f_3 - \frac{ab\pi^2}{6}\left(K - \omega^2\frac{\gamma h}{g}\right)\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right) f_6 = 0 \\ &-\frac{ab\pi^2}{6}\left(K - \omega^2\frac{\gamma h}{g}\right)\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right) + \left(K - \omega^2\frac{\gamma h}{g}\right) abf_6 = 0 \end{aligned}\right\} \tag{4.9}$$

where

$$A_2 = A - \frac{\omega^2\gamma h}{g}abJ = A - \frac{\omega^2\gamma h}{g}ab\left[\frac{\beta_1^2}{2} + \frac{\beta_2^2}{2} + \frac{1}{4} + \frac{\pi^4}{30}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right)\right.$$
$$\left. + \beta_1\beta_4\lambda^2 + \frac{\beta_2\beta_5}{\lambda^2} + \frac{\pi^4}{18}\beta_4\beta_5\right] \tag{4.10}$$

To find nonzero solution of f_3, f_6, we may obtain frequency equation

$$A - \frac{\omega^2\gamma hab}{g}J - \frac{ab\pi^4}{36}\left(K - \frac{\omega^2\gamma h}{g}\right)\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)^2 = 0 \tag{4.11}$$

Therefore we can find minimum eigenfrequency of free vibration of the thin plate as follows

$$\omega_{\min} = \left\{\frac{A - (abK\pi^4/36)(\beta_4\lambda^2 + \beta_5/\lambda^2)^2}{J - (\pi^4/36)(\beta_4\lambda^2 + \beta_5/\lambda^2)^2}\cdot\frac{g}{\gamma hab}\right\}^{\frac{1}{2}} \tag{4.12}$$

where

$$\left.\begin{aligned}
&A - \frac{abK\pi^4}{36}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)^2 = \frac{2D\pi^4}{l^2\lambda}\left[4(\beta_1^2 + \lambda^4\beta_2^2) + 2(1+\lambda^2)^2\right. \\
&\quad \left. + 2\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right) + \mu\beta_4\beta_5\lambda^2\right] + K\left[\frac{1}{2}\left(\beta_1^2 + \beta_2^2 + \frac{1}{2}\right)\right. \\
&\quad \left. + \frac{\pi^4}{180}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right) + \beta_1\beta_4\lambda^2 + \frac{\beta_2\beta_5}{\lambda^2}\right]ab \\
&J - \frac{\pi^4}{36}\left(\beta_4\lambda^2 + \frac{\beta_5}{\lambda^2}\right)^2 = \frac{\beta_1^2}{2} + \frac{\beta_2^2}{2} + \frac{1}{4} + \frac{\pi^4}{180}\left(\beta_4^2\lambda^4 + \frac{\beta_5^2}{\lambda^4}\right. \\
&\quad \left. + 6\beta_1\beta_4\lambda^2 + \frac{6\beta_2\beta_5}{\lambda^2}\right)
\end{aligned}\right\} \tag{4.13}$$

V. Example

Suppose the square plate on elastic foundations is acted on by a concentrated force at the center of the plate, and $\mu = 0.167$, $\lambda = a/b = 1$, $Ka^4/D = 10^4$.

It follows from expression (2.8) that

$$\beta_1 = \beta_2 = -6.98802, \quad \beta_4 = \beta_5 = -11.97605 \tag{5.1}$$

(1) From expressions (3,6) – (3,8), we have

$$1943.4280 f_3 + 39.39956 f_6 = 74.07538 Pa^2/D, \quad 39.39956 f_3 + f_6 = 10^{-4} Pa^2/D \tag{5.2}$$

Hence we find

$$f_3 = 0.08866 \times 10^{-4} Pa^2/D, \quad f_6 = -2.49323 \times 10^{-4} Pa^2/D$$

and from expression (2.9) we can count maximum deflection of the center of the plate

$$w(a/2, \ a/2) = 4.07440 \times 10^{-4} Pa^2/D \tag{5.3}$$

and prove antiforce $R_f = P$ without difficulty.

(2) Supposing $P_x = P_y$, $\gamma^0 = 1$, from expression (4.4) or (4.5) we may find minimum critical load is

$$(P_{cr})_{min}=\frac{391.10310}{17869.3120}Ka^2=\frac{391.10310}{17869.3126}\times10^4\frac{D}{a^2}=218.868\frac{D}{a^2} \tag{5.4}$$

If the square plate is acted on only by uniformly unilateral pressure $p_x=p$, then $p_y=0$ or $\gamma^0=0$, and so we may find minimum critical load is

$$(p_{cr})_{min}=437.736D/a^2 \tag{5.5}$$

(3) From expression (4.11) or (4.12) we can find minimum eigenfrequency

$$\omega_{min}=\left\{\frac{391.10310}{374.69269}\cdot\frac{Kg}{\gamma h}\right\}^{\frac{1}{2}}=\left\{10522.216\frac{gD}{\gamma ha^4}\right\}^{\frac{1}{2}}$$

$$=1.02570\times10^2\frac{1}{a^2}\sqrt{\frac{gD}{\gamma h}} \tag{5.6}$$

(4) Next we calculate moments of bending on free edges and at the center by paper [4]

$$M_x|_{x=0,a}=M_y|_{y=0,b}=-24.45913\times10^{-4}P \tag{5.7}$$

$$M_x(a/2,a/2)=M_y(a/2,a/2)=96.947036\times10^{-4}P \tag{5.8}$$

Obviously, moments of bending on free edges do not equal zero and the ratio of them with (5.8) is absolutely 0.25229. Hence they are not small quantities.

And we can calculate by this paper

$$M_x|_{x=0,a}=0,\quad M_y|_{y=0,b}=0$$

$$M_x(a/2,a/2)=M_y(a/2,a/2)=57.08753\times10^{-4}P \tag{5.9}$$

Thus the ratio of (5.8) and (5.9) is 1.69822, that is, (5.8) is nearly 70% larger than (5.9).

Because moments of bending on free edges do not equal zero but negative, the calculated results by paper [4] for moments of bending and maximum deflection at the center are that minimum critical load is larger, and eigenfrequency is smaller in comparison with the results of this paper.

VI. Conclusion

From the above discussion, because flexural function satisfies all boundary and corner conditions, and total antiforces of foundations equal total outer loads, the method and results of the paper are reliable, and the method is simple, by which we can find approximation solutions of bending, stability and vibrations of elastic thin plate with free edges on elastic foundations.

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