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Quantum enigma hidden in continuum mechanics*

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Abstract It is reported that there exist deformable media which display quantum effects just as quantum entities do. As such, each quantum entity usually treated as a point particle may be represented by a deformable medium, the dynamic behavior of which is prescribed by four dynamic state variables, including mass density, velocity, internal pressure, and intrinsic angular momentum. In conjunction with the finding of the characteristic equation characterizing the physical nature of such media, it is found that a complex field quantity may be introduced to uncover a perhaps unexpected correlation, i.e., the governing dynamic equations for such media may be exactly reduced to the Schrödinger equation, from which the closed-form solutions for all the four dynamic state variables can be obtained. It turns out that this complex field quantity is just the wavefunction in the Schrödinger equation. Moreover, the dynamic effects peculiar to spin are derivable as direct consequences. It appears that these results provide a missing link in quantum theory, in the sense of disclosing the physical origin and nature of both the wavefunction and the wave equation. Now, the inherent indeterminacy in quantum theory may be rendered irrelevant. The consequences are explained for certain long-standing fundamental issues.

Key words quantum entity, deformable medium, nonlinear dynamic equation, new interpretation, Schrödinger equation

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1 Introduction

It is well-known that, as the governing equation for the dynamic behavior of each quantum entity at subatomic scale, the Schrödinger equation should replace the Newton equation. Irrespective of the fact that the applicability and accuracy of quantum mechanics have been well established by numerous experiments and applications, it appears that there is no general consensus to what it really implies and what principles underlie it. The main reason behind this may be that, unlike other fundamental theories such as the theory of electromagnetism and the theory of relativity, the basic state variable in quantum theory is a complex field quantity, i.e., the wavefunction, which should be associated with the indeterminacy of probabilistic nature for the dynamic behavior of each quantum entity.

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Since the inception of quantum theory between 1925 and 1926, there have been various interpretations[1] centered on the meaning and the nature of the wavefunction, e.g., the orthodox Copenhagen interpretation, the hidden variable interpretation (cf., e.g., Bohm[5], Bohm and Hiley[6], and the newest development by Valentini[7]), the many worlds interpretation (cf., e.g., Everett[8], DeWitt and Graham[9], and Deutsch[10]), the ensemble interpretation (cf., e.g., Ballentine[11]), and those based on either decoherent histories (cf., e.g., Zeh[12], Zurek[13], Hartle[14], Gell-Mann[15], Gell-Mann and Hartle[16], Griffiths[17], and Schlosshauer[18]) or the spontaneous localization (cf., e.g., Ghirardi et al.[19] and Bassi and Ghirardi[20]) or quantum gravity (cf., e.g., Crane[21] and Penrose[22–23]). Recently, a deterministic theory underlying quantum mechanics has been proposed by t'Hooft[24]. However, it appears that the fundamental differences persist and sometimes render a clear understanding almost implausible. Regarding this puzzling situation, Feynman[25] remarked, "I think I can safely say that nobody understands quantum mechanics."

Since the efficacy of quantum theory in numerous applications is beyond any doubt[2], it is realized that a missing link is to be found toward completing the whole story[3]. For this purpose, however, first of all, it needs to unravel the perplexing issues centered on the physical origin and nature of both the complex wavefunction and the Schrödinger equation.

To uncover the missing link mentioned above, it is reported that there exist deformable media which display quantum effects just as quantum entities do. As such, each quantum entity treated as a point-like particle in quantum mechanics may be represented by a three-dimensional continuous medium moving and deforming in conservative force fields, the dynamic behavior of which is prescribed by four dynamic field variables, including a mass density field, a velocity field, a pressure field, and an intrinsic angular momentum field. In conjunction with the finding of the characteristic equation characterizing the physical nature of such media, the central result is the finding of a perhaps unexpected correlation, i.e., the Schrödinger equation can be derived from the governing equations for the deformable media. It will be demonstrated that a complex field quantity may be introduced to achieve an exact reduction of the nonlinear dynamic equations governing the deformable media to the Schrödinger equation, from which the closed-form solutions for all the dynamic variables can be obtained. It turns out that the introduced complex field quantity is nothing else but the wavefunction in the Schrödinger equation. With the uncovered correlation, it will further be shown that the direct physical quantities of the deformable medium at issue, such as the mass center, the momentum, the angular momentum, the intrinsic angular momentum, and Hamiltonian, agree with the corresponding expectation values in quantum theory. Moreover, new interpretations will be introduced to unravel certain long-standing fundamental issues.

The main content is arranged as follows. The dynamic state variables and the governing equations for the deformable media are introduced to represent the quantum entities in Section 2. In Section 3, with a characteristic equation and by means of a complex field quantity, the nonlinear dynamic equations governing the deformable media are exactly reduced to the Schrödinger equation, from which the closed-form solutions for all the dynamic state variables are obtained. It will be demonstrated in Section 4 that the direct physical quantities of the deformable media agree with the corresponding expectation values in quantum theory. In Section 5, the consequences with new interpretations are discussed for certain fundamental issues. In Section 6, the results are derived for the composite systems with multiple interacting quantum entities. Finally, the remarks are given in Section 7, and the derivations and proofs

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[1]Refer to, e.g., Mehra and Rechenberg[1] for a detailed account of the historical development of quantum theory. For the philosophy and the interpretation of quantum mechanics from various standpoints, references may be made to Healey[2], Omnès[3], and Weinberg[4].

[2]Refer to Weinberg[4], Section 4.

[3]Refer to Smolin[26], Section 1.

of the certain relevant results are given in Appendices A, B, and C.

Some notations that will be used are explained as follows. Let ∇ be the formal differential vector of the form $\nabla = \boldsymbol{e}_i \frac{\partial}{\partial x_i}$, where $\boldsymbol{e}_i$ and x_i (i = 1, 2, 3) are three orthonormal vectors and the three Cartesian coordinates of the spatial point $\boldsymbol{x}$. ∇^2 is the Laplacian. As usual, the gradient of the scalar field $f = f(\mathbf{x}, \mathbf{t})$ is a vector field denoted by ∇f, and the gradient and the divergence of the vector field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ are a second-order tensor field and a scalar field denoted by $\nabla \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$, respectively, which are given by

$$\nabla f = \frac{\partial f}{\partial x_i}\,\boldsymbol{e}_i, \quad \nabla \boldsymbol{u} = \frac{\partial u_j}{\partial x_i}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \nabla \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i}.$$

In the above, the repeated indices mean summation. Moreover, the notation $\oint (\cdots)\mathrm{d}V$ will be used to designate the integration over the entire space.

2 Deformable media representing quantum entities

In quantum theory, each quantum entity is represented by a point particle in an unusual sense different essentially from a classical point particle. Its dynamic behavior is so perplexing and elusive that it is beyond any usual experience. In particular, unlike a usual point particle, each such particle is so unusual that it has to be inextricably associated with a complex field quantity toward combining two extremes, i.e., the particle-like property and the wave-like property. As a consequence, at any given instant, this field quantity prescribes a complex value at every spatial point and all such values distributing over the whole space are assigned to the particle by introducing the probabilistic interpretation, thus leading to the inherent indeterminacy of probabilistic nature.

It seems that confusion and ambiguities that persist in quantum theory might be rooted in the very notion of the point particle, considering the fact that the field variable is fundamental but could not be assigned to a point particle in a natural way. We are going to show that a possible solution is to replace the notion of particles in any sense whatever with the notion of deformable media. On contrary to a point particle with concentrated mass and charge at a spatial point, a deformable medium is a physical entity with continuously distributing mass and charge in space. Such spatial distributions of different nature give rise to various shapes or configurations of the medium. The shape will change with the time-dependent fields under the force field and the surrounding conditions.

Here, the notion of the deformable medium is brought into focus on account of the facts as follows:

(i) It is noted that the field variable is fundamental in quantum theory, and a deformable medium is just a natural site carrying field variables.

(ii) The dynamic behavior of a deformable medium is fully determinate by prescribing certain dynamic state variables, thus bypassing the probabilistic indeterminacy associated with the notion of the point particle.

(iii) The possible singularities induced by the notion of the point particle will not be involved.

2.1 Deformable media with intrinsic angular momenta

With the foregoing facts in mind, each quantum entity will be represented by a continuous medium moving and deforming in the conservative force fields. How it moves and changes its shape depends on the physical nature of the medium itself, the nature of the applied conservative force field, the conditions in the surrounding, etc. It may be particle-like in a certain sense, while it may be wave-like in another sense. This initial understanding suggests that the wave-particle duality may be two particular aspects of the dynamic behavior of a deformable continuous medium.

The dynamic behavior of the deformable medium at issue is prescribed by certain dynamic state variables distributing over the entire space, i.e., each such variable is a time-dependent field variable. Specifically, consider a deformable medium of the total mass m moving and deforming subjected to both a conservative force field and a torque field. The force field is given by $-\frac{1}{m}\nabla U$ per unit mass[4], where

$$U = U(\boldsymbol{x}, t) \tag{1}$$

is the potential energy function, while the torque field[5] per unit mass is given by

$$\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{x}, t). \tag{2}$$

The dynamic behavior of the deformable medium at issue is prescribed by four dynamic variables, including a normalized mass density field, a velocity field, an internal pressure field, and an intrinsic angular momentum field, represented by ρ, $\boldsymbol{v}$, q, and $\boldsymbol{\alpha}$, respectively. Each of the four dynamic variables is a function of the spatial position $\boldsymbol{x}$ and the time t, i.e.,

$$\begin{cases} \rho = \rho(\boldsymbol{x}, t), \quad \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t), \\ q = q(\boldsymbol{x}, t), \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{x}, t). \end{cases} \tag{3}$$

Note that the usual mass density is given by $m\rho$. Thus, the normalized condition is

$$\oint \rho \mathrm{d}V = 1. \tag{4}$$

Here, the use of ρ is for the purpose of conforming to the standard form of the Schrödinger equation (cf., Footnote 2).

It is noted that the four field variables in Eq. (3) represent the dynamic behavior of a deformable medium in a natural and direct sense. In fact, ρ specifies the inertia distribution of the medium, $\boldsymbol{v}$ is the velocity of each infinitesimal mass element at the spatial point $\boldsymbol{x}$ and at the instant t, q characterizes the internal interaction[6] of the medium as a physical entity in response to the shape changing, and the field $\boldsymbol{\alpha}$ provides the angular momentum per unit mass at each spatial point and emerges in response to the applied torque field $\boldsymbol{\beta}$.

2.2 Governing equations for dynamic state variables

At each instant t, each infinitesimal mass element of a deformable medium observes three fundamental principles[7], i.e., the mass conservation principle, the momentum balance principle, and the angular momentum balance principle. From these principles, the field equations governing the state variables introduced in the last subsection may be derived, as is shortly done below.

At the instant t and the spatial point $\boldsymbol{x}$, an infinitesimal mass element is given by $m\rho\mathrm{d}V$, where $\mathrm{d}V$ is the instantaneous volume of the mass element at issue. The mass conservation principle requires that the time derivative of this infinitesimal mass should vanish. From this and the formula

$$\dot{\overline{\mathrm{d}V}} = (\nabla \cdot \boldsymbol{v})\mathrm{d}V, \tag{5}$$

we have the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{6}$$

[4]Here, the factor $\frac{1}{m}$ is introduced for the purpose of conforming to the standard form of the Schrödinger equation (see Eq. (19) in the next section).

[5]This will be induced in a moving medium with charge in a magnetic field.

[6]In a broad sense, there arises a stress tensor field in a deformable medium. In the sequel, it will suffice to consider the simplest case when the stress field is given by the all-around pressure of the form $q\boldsymbol{I}$ (here, $\boldsymbol{I}$ is the second-order identity tensor).

[7]Here, we assume that the motion is nonrelativistic, i.e., $|\boldsymbol{v}| \ll c$.

Next, the linear momentum of the foregoing mass element is given by $(m\rho \mathrm{d}V)\boldsymbol{v}$, while the resultant force acting on this mass element is given by $(\nabla q)\mathrm{d}V-(\rho\nabla U)\mathrm{d}V$. Then, the momentum balance principle produces the Cauchy equation of motion as follows:

$$\nabla q-\rho\nabla U=m\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot(\nabla\boldsymbol{v})\Big). \tag{7}$$

Moreover, the angular momentum and the resultant moment of forces for the mass element at issue are given by

$$m\rho\boldsymbol{\alpha}\mathrm{d}V+\boldsymbol{x}\times(m\rho\boldsymbol{v}\mathrm{d}V),\quad \boldsymbol{x}\times(\nabla q-\rho\,\nabla U)\mathrm{d}V+m\rho\boldsymbol{\beta}\mathrm{d}V.$$

With these and Eq. (7), the angular momentum balance principle yields

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}=\boldsymbol{\beta}. \tag{8}$$

2.3 Characteristic equation for quantum entities

The nonlinear dynamic equations (6)–(8) are derived from three universal physical principles and common to all deformable media. In addition to these general equations, it may be essential to present a characteristic equation representing the physical nature unique to a deformable medium in response to every shape change it possibly undergoes. Generally, this equation prescribes how the stress relates to every deformation a deformable medium may experience and, in particular, how the internal pressure q relates to the density ρ. It will play a central role in the whole development. This will further be discussed in Section 3 and Section 5.

2.4 Conditions at infinity

Each dynamic variable should vanish at infinity, i.e.,

$$\begin{cases}\lim\limits_{|\boldsymbol{x}|\to\infty}\rho=0, & \lim\limits_{|\boldsymbol{x}|\to\infty}q=0,\\[2ex] \lim\limits_{|\boldsymbol{x}|\to\infty}\boldsymbol{v}=\mathbf{0}, & \lim\limits_{|\boldsymbol{x}|\to\infty}\rho\boldsymbol{\alpha}=\mathbf{0}.\end{cases} \tag{9}$$

Note that the last equation requires that the intrinsic angular momentum of any volume element at infinity vanish.

3 Derivation of Schrödinger equation

The idea of replacing the notion of the point particle with the notion of the deformable medium in representing the quantum entities may be meaningful only in the case when the well-established results in quantum theory may be incorporated and, furthermore, the new results missing in quantum theory may be disclosed.

Toward the above goal, the central issue is to find out the characteristic equation for deformable quantum entities and then work out the explicit solutions for the four dynamic state variables as listed in Eq. (3). At the first sight, it may not be feasible to resolve the nonlinear dynamic problem of a deformable medium subjected to a force field in a general sense. In fact, Eqs. (6) and (7) form a complicated and coupled system of four highly nonlinear differential equations for two scalar fields and a vector field, i.e., ρ, q and $\boldsymbol{v}$. Moreover, the fact that the characteristic equation indicated in Subsection 2.3 is to be found may further complicate the situation. On account of these undue complexities, generally, it would be difficult to obtain any non-trivial results of clear physical meanings from Eqs. (6)–(9). Moreover, from a physical standpoint, a question may arise, i.e., how any quantum effect characteristics of the quantum entities could be derived from Eqs. (6)–(9) for the deformable media. Indeed, usually, each field variable of the deformable medium is continuously differentiable, and, as such, could not be related to any quantized phenomena. In particular, it may be quite puzzling that how a deformable medium subjected to a potential field U (cf., Eq. (1)) could display both the wave-like property and the particle-like property.

If the challenges indicated above would prove to be intractable, the idea of representing quantum entities by deformable continuous media would be merely of formal sense. To answer these challenges means that we need to seek for unusual deformable media which display quantum effects just as quantum entities do. This relies on whether the characteristic equation for such media may be found out, in the sense that the Schrödinger equation is derivable as a direct consequence. The answer will be given below.

3.1 Finding of characteristic equation

As indicated before, it is essential to find out the characteristic equation which represents the physical nature of each quantum entity in response to the shape changing. Since the quantum entities are characterized just by the Schrödinger equation, the form of the characteristic equation for the quantum entities should be such that the Schrödinger equation will eventually be derived as a consequence. This equation is found to be of the form

$$q = \frac{\hbar^2}{4m}\rho\nabla^2(\ln\rho), \tag{10}$$

where $\hbar = 1.054\,57 \times 10^{-34}$ J·s is Planck's constant, and m is the mass of the medium. Equation (10) represents the physical nature of a quantum entity in response to the shape changing. It is the finding of Eq. (10) that makes it possible to uncover the correlation to the Schrödinger equation, as will be shown below.

3.2 Closed-form solutions for $\boldsymbol{\rho}$ and $\boldsymbol{v}$

Now, Eqs. (6), (7), and (10) form a closed system of nonlinear dynamic equations, from which the density field ρ, the velocity field $\boldsymbol{v}$, and the pressure field q may be determined under any given force field. It is found that, by introducing two field variables as follows:

$$R = R(\boldsymbol{x}, t), \quad S = S(\boldsymbol{x}, t), \tag{11}$$

the closed-form solutions for ρ and $\boldsymbol{v}$ may be presented as follows:

$$\rho = R^2 + S^2, \tag{12}$$

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla\Big(\arctan\Big(\frac{S}{R}\Big)\Big). \tag{13}$$

The field variables R and S will be governed by two equations derived in the next subsection. Their meanings will be indicated in the subsequent development.

3.3 Equations governing $\boldsymbol{R}$ and $\boldsymbol{S}$

The governing equations for R and S are derived by substituting Eqs. (10), (12), and (13) into Eqs. (6) and (7). The results are

$$\hbar\Big(R\frac{\partial R}{\partial t} + S\,\frac{\partial S}{\partial t}\Big) = -\frac{\hbar^2}{2m}(R\nabla^2 S - S\nabla^2 R), \tag{14}$$

$$\hbar\Big(S\frac{\partial R}{\partial t} - R\frac{\partial S}{\partial t}\Big) = -\frac{\hbar^2}{2m}(R\nabla^2 R + S\nabla^2 S) + \rho U. \tag{15}$$

The details may be found in Appendix A.

3.4 Reduction to Schrödinger equation

By performing the following procedures:

$$R \times \text{Eq. (14)} + S \times \text{Eq. (15)}, \quad S \times \text{Eq. (14)} - R \times \text{Eq. (15)},$$

using Eq. (12), and then eliminating ρ, Eqs. (14) and (15) may be recast as follows:

$$\hbar\frac{\partial R}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 S + SU, \tag{16}$$

$$\hbar\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\nabla^2 R - RU. \tag{17}$$

Then, by introducing the following complex field quantity:

$$\Psi = R + \mathrm{i}S \tag{18}$$

and performing the following procedure:

$$\mathrm{i} \times \text{Eq. (16)} - \text{Eq. (17)},$$

Eqs. (16) and (17) may be converted to

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi + U\Psi. \tag{19}$$

This linear differential equation for the complex field quantity Ψ is just the Schrödinger equation, and Ψ is just the wavefunction.

The wavefunction Ψ is determined from the Schrödinger equation (19) with an initial condition, the normalized condition Eq. (4), and the condition at infinity (cf., Eq. (9)), i.e.,

$$\oint \Psi^*\Psi \mathrm{d}V = 1, \tag{20}$$

$$\lim_{|\boldsymbol{x}|\to\infty} \Psi = 0. \tag{21}$$

3.5 Direct expressions in terms of wavefunction

Equation (10) for the pressure field q and Eqs. (14) and (15) for the density field ρ and the velocity field $\boldsymbol{v}$ may be presented directly in terms of the wavefunction Ψ, i.e.,

$$\rho = \Psi^*\Psi, \tag{22}$$

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla\Big(\arctan\Big(\mathrm{i}\frac{\Psi^* - \Psi}{\Psi^* + \Psi}\Big)\Big), \tag{23}$$

$$q = \frac{\hbar^2}{4m}(\Psi^*\Psi)\nabla^2(\ln(\Psi^*\Psi)), \tag{24}$$

where Ψ^* is the complex conjugate of Ψ.

It is clear that the Schrödinger equation (19) with Eqs. (20) and (21) determines the wavefunction Ψ and then all the state variables via Eqs. (22)–(24).

3.6 Intrinsic angular momentum

Since there is no coupling between Eq. (7) and Eqs. (5) and (6), the intrinsic angular momentum $\boldsymbol{\alpha}$ per mit mass is independent of the Schrödinger equation. Therefore, it may be treated in a decoupled manner. In the absence of magnetic field, the torque field $\boldsymbol{\beta}$ (cf., Eq. (2)) in Eq. (8) becomes vanishing. Then, Eq. (8) can produce a constant intrinsic angular momentum field $\boldsymbol{\alpha}$ as follows:

$$\boldsymbol{\alpha} = \sqrt{s(s+1)}\frac{\hbar}{m}\boldsymbol{\omega}, \tag{25}$$

where s is a constant number, and $\boldsymbol{\omega}$ is a constant unit vector. It should be noted that $\boldsymbol{\alpha}$ needs not vanish in the absence of magnetic field. It is an intrinsic property and related to the notion of the spin, as will be discussed in the next section.

In the presence of magnetic field, denoted B, the unit vector $\boldsymbol{\omega}$ in Eq. (25) should be determined from Eq. (8) with $\beta = g\alpha \times B$, where g is a constant factor.

3.7 Dynamic features of quantum entities

The crucial step in the above is the finding of the characteristic equation (10) representing the physical nature of the quantum entities as deformable media. It has been demonstrated that the Schrödinger equation (19) is derivable as a consequence from this equation. It may be further demonstrated that the opposite may also be true. Namely, if the density field ρ of a deformable medium is given by Eq. (12) with the complex field variable Ψ therein governed by the Schrödinger equation (19), the characteristic equation of this deformable medium should be given by Eq. (10). In fact, Eq. (19) leads to Eqs. (14) and (15). From Eqs. (6), (12), (14), and (A1) in Appendix A, we can derive Eq. (A3). Then, from Eqs. (A3) and (A2), we can obtain the velocity field $\boldsymbol{v}$ given by Eq. (13). Furthermore, with Eqs. (12) and (13), we may convert Eq. (6) to Eq. (A14), as will be shown in Appendix A. Thus, from Eq. (A14) and Eq. (15), we can derive Eq. (10).

Moreover, from the reduction and correlation disclosed, it may follow that the dynamic behavior of the deformable medium representing each quantum entity is eventually determined by the wavefunction Ψ (cf., Eq. (18) with Eq. (11)) via the Schrödinger equation (19). This answers the challenges indicated at the outset of this section. In fact, from the correlation uncovered, it follows that all quantum effects governed by the Schrödinger equation are indeed incorporated into the dynamic behavior of the deformable medium represented by the general equations (6) and (7) and the characteristic equation (10). Furthermore, the spin angular momentum, which is not governed by the Schrödinger equation, is also incorporated into Eq. (8). From these facts, it may be concluded that each quantum entity turns out to be a deformable medium with the characteristic equation (10).

Equations (22)–(24) clearly show the dynamic features of each quantum entity in a force field. Indeed, the Schrödinger equation (19) determines the complex wavefunction Ψ evolving with time. Then, the amplitude of Ψ (cf., Eq. (42) given later) determines the density field and the pressure field via Eqs. (22) and (24), while the gradient of the phase angle of Ψ specifies the velocity field via Eq. (23) (cf., Eq. (43) given later). As such, generally, each field is in constant change in a force field. The stationary states with a time-independent amplitude of Ψ and the time-dependent but uniformly distributing phase angle of Ψ are of particular interest. In this case, both the density field and the pressure field are time-independent, and, moreover, the velocity field vanishes.

4 Relations to observables in quantum theory

In quantum theory based on the notion of point particles, the expectation values of the observables are calculated by prescribing certain ad hoc mathematical manipulations via the Hermitian operators. Each of them is assumed to supply the estimate of a physical quantity from a statistical standpoint. However, it may not be clear whether or not each such estimate of ad hoc nature is really of physical relevance. Now, with the new notion of the deformable medium, various physical quantities may be given in a natural and direct sense. In this section, we are going to show that certain basic physical quantities agree with the expectation values in quantum theory. This will be done for the position, the linear momentum, the angular momentum, the Hamiltonian, the spin angular momentum, etc.

4.1 Mass center and linear momentum

The mass center and the linear momentum of the medium are denoted by $\boldsymbol{x}_0$ and $\boldsymbol{p}_0$, and defined by

$$\boldsymbol{x}_0 = \frac{\oint m\rho\boldsymbol{x}\mathrm{d}V}{\oint m\rho\mathrm{d}V} = \oint \rho\boldsymbol{x}\mathrm{d}V, \tag{26}$$

$$\boldsymbol{p}_0 = \oint m\rho\boldsymbol{v}\mathrm{d}V = m\oint \rho\boldsymbol{v}\mathrm{d}V. \tag{27}$$

According to quantum theory, the expectation values[8] of the position and the momentum are

$$\langle \boldsymbol{x} \rangle = \oint \boldsymbol{x} \Psi^* \Psi \mathrm{d}V, \tag{28}$$

$$\langle \boldsymbol{p} \rangle = -\mathrm{i}\hbar \oint \Psi^* \nabla \Psi \mathrm{d}V. \tag{29}$$

From Eqs. (22), (26), and (28), it follows

$$\langle \boldsymbol{x} \rangle = \boldsymbol{x}_0. \tag{30}$$

Moreover, substituting Eq. (18) into Eq. (29) yields

$$\langle \boldsymbol{p} \rangle = \hbar \oint (R \nabla S - S \nabla R) \mathrm{d}V - \frac{1}{2} \mathrm{i}\hbar \oint \nabla \rho \mathrm{d}V. \tag{31}$$

The second integration at the above right-hand side may be shown to vanish by use of the Gauss theorem and Eq. $(9)_1$. Hence, from this fact and Eq. (A2) in Appendix A, we can infer

$$\langle \boldsymbol{p} \rangle = \boldsymbol{p}_0. \tag{32}$$

4.2 Angular momentum and spin angular momentum

By following the same procedures, we can prove that the angular momentum of the medium

$$\boldsymbol{M}_0 = \oint \boldsymbol{x} \times m\rho \boldsymbol{v} \mathrm{d}V \tag{33}$$

agrees with the expectation value

$$\langle \boldsymbol{M} \rangle = -\mathrm{i}\hbar \oint \Psi^* (\boldsymbol{x} \times \nabla \Psi) \mathrm{d}V, \tag{34}$$

i.e.,

$$\langle \boldsymbol{M} \rangle = \boldsymbol{M}_0. \tag{35}$$

On the other side, the intrinsic angular momentum of the medium is given by

$$\boldsymbol{S}_0 = \oint m\rho \boldsymbol{\alpha} \mathrm{d}V = \sqrt{s(s+1)} \hbar \boldsymbol{\omega}. \tag{36}$$

In deriving the above, Eqs. (4) and (25) have been used. It is an intrinsic property that every elementary "particle" has a specific and immutable value of s known as the spin. For instance, $s = \frac{1}{2}$ for electrons, neutrons, protons, etc. The magnitude of the intrinsic angular momentum $\boldsymbol{S}_0$ given by Eq. (36) agrees with the countpart in quantum theory.

It may be noted that the spin angular momentum $\boldsymbol{S}_0$ is independent, and hence could not be derived from the Schrödinger equation. However, as a direct consequence of the intrinsic angular momentum per unit mass, it is naturally incorporated into the dynamic behavior of the deformable medium.

[8]Formulas for the expectation values of the observables may be found in the standard monographs for quantum mechanics, e.g., earlier in Dirac[27] and recently in Griffiths[28].

4.3 Hamiltonian

As the total energy, the Hamiltonian of the medium is the sum of the kinetic energy, the potential energy, and the work done by the internal pressure q (denoted by W_0), i.e.,

$$H_0 = W_0 + \oint \Big(\rho U + \frac{1}{2} m\rho |\boldsymbol{v}|^2\Big) \mathrm{d}V, \tag{37}$$

where the work done by the pressure q is (cf., Eq. (5))

$$W_0 = \int_0^t \Big(\oint q \frac{\dot{\overline{\mathrm{d}V}}}{\mathrm{d}V} \mathrm{d}V\Big) \mathrm{d}t = \int_0^t \Big(\oint q \nabla \cdot \boldsymbol{v} \mathrm{d}V\Big) \mathrm{d}t. \tag{38}$$

The above integration may be worked out to produce (see Appendix B)

$$W_0 = \frac{\hbar^2}{8m} \oint \rho |\nabla(\ln\rho)|^2 \mathrm{d}V. \tag{39}$$

The origin of the Hamiltonian, i.e., the total energy, is clear. In addition to the kinetic energy and the potential energy, the work done by the internal pressure q in response to the changing shape will be stored in the medium and contributes to the total energy.

According to quantum theory, on the other hand, the expectation value of the Hamiltonian is given by

$$\langle H \rangle = \oint \Psi^* \Big(-\frac{\hbar^2}{2m} \nabla^2 \Psi + U\Psi\Big) \mathrm{d}V. \tag{40}$$

It may be proven (see Appendix C) that Eq. (37) and Eq. (40) agree with each other, i.e.,

$$\langle H \rangle = H_0. \tag{41}$$

Remark 1 Each established relation does not mean agreement in physical content. In fact, with the notion of the deformable medium, each expectation value is no longer of physical relevance.

5 New interpretations

Based on the notion of the deformable medium, new interpretations may be introduced to explain and understand the dynamic behavior of each quantum entity. Now, both the results and the understanding based on the notion of the point particle need to be either changed or reinterpreted. In what follows, the consequences will be discussed for certain long-standing fundamental issues.

5.1 Nature of wavefunction and Schrödinger equation

The physical origin and meaning of the wavefunction and Schrödinger equation may be among the most mysterious. Questions in two respects may be essential. On the one side, how and why can a complex field quantity, i.e., the wavefunction Ψ (cf., Eq. (18)), serve as a basic variable? On the other side, how does the Schrödinger equation come into play as a fundamental equation and what principles underlie it?

It turns out that the answer to the above questions lies in the correlation uncovered in Section 3, i.e., the coupled system of the five dynamic equations in Eqs. (6), (7), and (10) governing the deformable media are exactly reducible to a single linear differential equation and the latter is just the Schrödinger equation (19), and it is the complex field quantity Ψ (cf., Eqs. (11) and (18)), i.e., the wavefunction, that renders this correlation possible.

Now, it becomes clear that the Schrödinger equation (19) is just the reduced linear differential equation derived from the nonlinear dynamic equations (6), (7), and (10). Behind this is the physical nature unique to each quantum entity, i.e., the characteristic equation representing the physical nature of the quantum entities in response to the shape changing (see Subsections 2.3 and 3.1 and in particular Eq. (10)) is such that Eqs. (6) and (7) can be reduced to a single linear differential equation, i.e., the Schrödinger equation. Generally, it may be thought that the essential complexities from both mathematical and physical standpoints may be introduced by replacing the notion of the point-like particle with the notion of the deformable medium. It is found that the dynamic behavior of the quantum entities as the deformable media may be exactly reduced to a linear problem based on a complex field quantity via the Schrödinger equation.

The meaning of the wavefunction Ψ (cf., Eq. (18)) with the two field variables R and S (cf., Eq. (11)) may be derived from Eqs. (12) and (13). In fact, by virtue of Euler's formula, the wavefunction Ψ may be given by

$$\begin{cases} \Psi = R + \mathrm{i}S = \psi \mathrm{e}^{\mathrm{i}\vartheta}, \quad \psi = \sqrt{\rho} = \sqrt{R^2 + S^2}, \\ \vartheta = \arctan \dfrac{S}{R}, \quad R = \psi \sin\vartheta, \quad S = \psi \cos\vartheta, \end{cases} \tag{42}$$

where ψ and ϑ are the amplitude and the phase angle of the wavefunction Ψ, respectively:

$$\psi = \psi(\boldsymbol{x}, t), \quad \vartheta = \vartheta(\boldsymbol{x}, t).$$

Then, from Eq. (23), the velocity field may be alternatively given by

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla\vartheta, \tag{43}$$

and, therefore,

$$\frac{\hbar}{m}\nabla^2\vartheta = \nabla\cdot\boldsymbol{v} = -\frac{\mathrm{d}(\ln\rho)}{\mathrm{d}t}. \tag{44}$$

Thus, it follows that the amplitude of Ψ is specified by the density field ρ, while the phase angle of Ψ is determined by the changing rate of the logarithmic density. Moreover, the two field variables R and S are prescribed by ρ and ϑ, respectively, as shown in Eqs. (42).

A further meaning will be indicated in the next subsection.

5.2 Origin of quantized phenomena

Arising from the nonlinear features of the governing equations (6)–(9) with a given conservative force field U, infinitely many solutions of discretized nature for the state variables may be derived from these equations under the vanishing conditions (see Eq. (9)) at infinity. Then, the discretized values of a physical quantity of the medium may be obtained from such discretized solutions (see Section 4). However, it appears difficult to derive such solutions directly from Eqs. (6), (7), and (10) with Eq. (9). It is based on the Schrödinger equation that this significant issue may be reduced to working out the eigenvalues and eigenfunctions of the stationary states or determinate states in general, as has been done in quantum theory.

5.3 On probabilistic interpretation

Now, the dynamic behavior of each quantum entity as a deformable medium is completely prescribed by the four field variables (cf., Eq. (3)), i.e., the density field ρ, the velocity field $\boldsymbol{v}$, the pressure field q, and the intrinsic angular momentum field $\boldsymbol{\alpha}$. Each of them (cf., Eqs. (23)–(25)) is determined by the wavefunction Ψ via Eq. (19). As a consequence, the inherent indeterminacy associated with the notion of the particle may be eliminated with the central role of the Schrödinger equation in a new sense based on the notion of the deformable medium.

It may be clear that the probabilistic interpretation based on the notion of the point-like particle would become irrelevant. It should be noted that the field variable ρ is of the same value as in quantum theory but of different meaning. It is no longer the probability density at all but the normalized mass density of the medium.

5.4 On wave-particle duality

From Eqs. (22) and (42), we can see that the amplitude of the wavefunction Ψ prescribes the density field ρ, while the phase angle of the wavefunction determines the changing rate of ρ. The wavefunction Ψ governed by the Schrödinger equation is such that both the particle-like property and the wave-like property are combined into the density field ρ and the changing rate of ρ. Unlike the density field ρ, however, the phase angle ϑ does not need to be localized in a very narrow spatial region.

The double-slit experiment may also be explained in a direct and natural manner. Since each quantum entity is not a point particle but actually a continuous medium moving and deforming in space, there are always parts of this medium that arrive at and pass through the two slits, and such parts with changing phase angles may interfere with each other. In particular, infinitely many points of this medium pass through the two slits along infinitely many paths.

5.5 On completeness of quantum theory

The Schrödinger equation (19) prescribes all the dynamic state variables via the wavefunction Ψ (see Eqs. (22)–(24)). Every well-established result in quantum theory remains intact, as it should be, except for the fact that a new interpretation should be given based on the notion of the deformable medium.

5.6 On measurement problem

For the dynamic behavior of a quantum entity, now, non-uniform time-dependent fields need be treated. Each such field is varying from point to point in space at every instant. However, usual means of measurement can merely produce a single value each time, and may induce non-predictable appreciable disturbance to the spatial distribution of each field in the medium. As such, there arises uncertainty in two respects, i.e., (i) the value of a field quantity at each spatial point could not be accurately measured, and (ii) the spatial distribution of a field quantity could not be derived from the result at each measurement. In association with such uncertainties, the result at each measurement rests essentially on the unusual deformation nature as represented by Eq. (10).

6 Composite systems with interacting quantum entities

It is the objective of this section to extend the main idea and results in the preceding sections to every composite system consisting of multiple quantum entities interacting with one another. Typically, such a system is an atom consisting of a heavy nucleus, with the electric charge $z\mathrm{e}$, surrounded by z interacting electrons with the mass m and the charge $-\mathrm{e}$.

Generally, each such composite system is represented by z deformable continuous media interacting with one another. Associated with them are z spatial position vectors and z masses, i.e., $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z)$, $(m_1, \cdots, m_z)$, and the charges. The inertia distribution of such a composite system is given by a normalized density field described as follows:

$$\rho = \rho(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t) \tag{45}$$

with the normalized condition[9]

$$\oint \rho \mathrm{d}V_1 \cdots \mathrm{d}V_z = 1. \tag{46}$$

Here and henceforward, $\mathrm{d}V_1, \cdots, \mathrm{d}V_z$ are used to denote z infinitesimal volume elements associated with the z mutually interacting media. The mass densities for the latter are therefore given by $m_1\,\rho, \cdots, m_z\,\rho$, separately.

[9]Note here that the density ρ is defined in association with all the volume elements of the z interacting media. That will also be a case (see Eq. (52)) for the conservative force field $\nabla_k U$ and the equivalent force field $\nabla_k q$ arising from the pressure field q.

The interaction between these z media is characterized by an interacting potential function shown below:

$$U = U(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t). \tag{47}$$

The gradient of this potential with respect to the spatial variable $\boldsymbol{x}_k$ provides a force field per unit mass, which is associated with the medium k and given by $-m_k^{-1}\nabla_k U$ (see Footnote 2). Here and below, ∇_k is the differential vector with respect to the spatial variable $\boldsymbol{x}_k$.

On the other hand, there are z velocity fields associated with the z media designated by

$$\boldsymbol{v}_k = \boldsymbol{v}_k(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t), \quad k = 1, \cdots, z. \tag{48}$$

The velocity field of the composite system is available from the fact that the linear momentum of every infinitesimal mass element is just the sum of the linear momenta (see Eq. (53)) associated with the z media, as given below:

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t) = \frac{m_1}{m}\boldsymbol{v}_1 + \cdots + \frac{m_z}{m}\boldsymbol{v}_z, \tag{49}$$

where m is the total mass, and $m = m_1 + \cdots + m_z$.

As before, the pressure field is denoted by q and now of the form

$$q = q(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t). \tag{50}$$

The spatial gradient of q with respect to $\boldsymbol{x}_k$ will determine the force per unit mass acting on the medium k (see Eq. $(52)_2$ and Footnote 8 for details). Moreover, the intrinsic angular momentum for each medium may be given as before, and will no longer be discussed below.

As in Section 2, the mass conservation principle and the momentum balance principle give rise to the continuity equation and the equation of motion. Firstly, consider an infinitesimal mass element of the medium k with the mass given by (cf., Eqs. (45) and (46)) $m_k\,\rho \mathrm{d}V_1 \cdots \mathrm{d}V_z$. The mass conservation principle for the medium k requires that the time derivative of the above infinitesimal mass should vanish. From this, Eq. (45), and the equalities (cf., Eq. (5))

$$\dot{\overline{\mathrm{d}V_k}} = (\nabla_k \cdot \boldsymbol{v}_k)\mathrm{d}V_k, \quad k = 1, \cdots, z,$$

we can derive the continuity equation for the composite system as follows:

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{z} \nabla_k \cdot (\rho \boldsymbol{v}_k) = 0. \tag{51}$$

Next, the linear momentum and the resultant force acting on an infinitesimal mass element associated with the medium k are given by[10]

$$(m_k \rho \mathrm{d}V_1 \cdots \mathrm{d}V_z)\boldsymbol{v}_k, \quad (\nabla_k q - \rho\,\nabla_k U)\mathrm{d}V_1 \cdots \mathrm{d}V_z. \tag{52}$$

Then, from these and the linear momentum balance principle for each component medium, we can derive the following equation of motion:

$$\nabla_k q - \rho\,\nabla_k U = m_k \rho\Big(\frac{\partial \boldsymbol{v}_k}{\partial t} + \sum_{r=1}^{z} \boldsymbol{v}_r \cdot (\nabla_r \boldsymbol{v}_k)\Big), \quad k = 1, \cdots, z. \tag{53}$$

[10] According to the work principle for the deformable media, the equivalent force per unit mass resulting from the internal pressure q is given by the spatial gradient of q divided by the mass density.

Note that the content in the brackets on the right-hand side of Eq. $(53)_1$ is just the acceleration associated with the medium k.

The system of the differential equations formed by Eqs. (51) and (53) is of the same structure as that formed by Eqs. (6) and (7). Hence, the results may be derived following the same procedures in Section 3. In fact, by means of the following two field variables (cf., Eq. (11)):

$$R = R(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t), \quad S = S(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t), \tag{54}$$

the following solution may be presented (cf., Eqs. (10), (12), and (13)):

$$\begin{cases} \rho = R^2 + S^2, \quad q = \displaystyle\sum_{k=1}^{z} \frac{\hbar^2}{4m_k} \rho \nabla_k^2 (\ln \rho), \\ \boldsymbol{v}_k = \dfrac{\hbar}{m} \nabla_k \Big(\arctan \Big(\dfrac{S}{R} \Big) \Big), \quad k = 1, \cdots, z. \end{cases} \tag{55}$$

Then, by following the same procedures in Appendix A, Eqs. (51) and (53) may be reduced to the following two equations for R and S:

$$\begin{cases} \hbar \Big(R \dfrac{\partial R}{\partial t} + S \dfrac{\partial S}{\partial t} \Big) = -\displaystyle\sum_{k=1}^{z} \frac{\hbar^2}{2m_k} (R \nabla_k^2 S - S \nabla_k^2 R), \\ \hbar \Big(S \dfrac{\partial R}{\partial t} - R \dfrac{\partial S}{\partial t} \Big) = -\displaystyle\sum_{k=1}^{z} \frac{\hbar^2}{2m_k} (R \nabla_k^2 R + S \nabla_k^2 S) + \rho U. \end{cases} \tag{56}$$

Now, by introducing the following wavefunction:

$$\Psi = R + \mathrm{i} S = \Psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_z, t) \tag{57}$$

and applying the same procedures in treating Eqs. (14) and (15), we can infer that the continuity equation (51) and the motion equations (53) may be reduced to the Schrödinger equation for composite systems, i.e.,

$$\mathrm{i}\hbar \frac{\partial \Psi}{\partial t} = -\sum_{k=1}^{z} \frac{\hbar^2}{2m_k} \nabla_k^2 \Psi + U \Psi. \tag{58}$$

Thus, the normalized density field ρ, the velocity field $\boldsymbol{v}$ (cf., Eq. (49)), and the pressure field q for the composite system are expressed in terms of the wavefunction Ψ as follows:

$$\rho = \Psi^* \Psi, \tag{59}$$

$$\boldsymbol{v} = \frac{\hbar}{m} \sum_{k=1}^{z} \nabla_k \Big(\arctan \Big(\mathrm{i} \frac{\Psi^* - \Psi}{\Psi^* + \Psi} \Big) \Big), \tag{60}$$

$$q = \sum_{k=1}^{z} \frac{\hbar^2}{4m_k} (\Psi^* \Psi) \nabla_k^2 (\ln(\Psi^* \Psi)). \tag{61}$$

Evidently, the results for a single entity, as given in Section 3, are the particular case of the above general results with $z = 1$.

7 Conclusions

Various fundamental issues in quantum mechanics are centered on the inherent indeterminacy of probabilistic nature. It appears that the root of this indeterminacy is the assumption that the dynamic behavior associated with a complex field quantity, i.e., the wavefunction, should be interpreted based on the notion of point-like particles. It has been demonstrated that this indeterminacy and related issues may be rendered irrelevant by representing each quantum entity with a deformable medium. The main results are summarized as follows.

Each quantum entity is a continuous medium characterized by Eq. (10), moving and deforming in the conservative force fields, with the dynamic state variables ρ, $\boldsymbol{v}$, and q prescribed by the wavefunction Ψ via the Schrödinger equation (19), as given by Eqs. (22)–(24), as well as the the intrinsic angular momentum $\boldsymbol{\alpha}$ determined by Eq. (8), and in particular, Eq. (25). These results are the direct consequences of the finding of an unexpected correlation, i.e., the coupled system of the five nonlinear dynamic equations in Eqs. (6), (7), and (10) governing the deformable media are exactly reduced to a single linear differential equation, and the later is just the Schrödinger equation. The wavefunction Ψ is a complex field quantity that renders this reduction possible. Moreover, the spin angular momentum is incorporated in the dynamic behavior of the deformable medium.

The physical meaning of the wavefunction may accordingly be disclosed. Namely, its amplitude can be prescribed by the normalized density field ρ, while its phase angle can be determined by the changing rate of the logarithmic density, as indicated in Eqs. (42)–(44). Moreover, the origin of the total energy may be explained, i.e., it is the sum of the stress work stored in the medium, the kinetic energy, and the potential energy.

As contrasted with the notion of the point particle, the notion of the deformable medium substantially broadens the scope of the dynamic effects, in a sense that both the shape changes and the spatial distributions are allowed to come into play. It may be expected that this may play roles both in understanding the observed phenomena and in envisaging and discovering new phenomena, in particular, in bypassing possible singularities. Such singularities, which typically emerge in an effort to combine the general theory of relativity at large scale and the quantum theory at microscopic scale, are rooted in the very notion of point particles. In fact, a zero-dimensional particle without spatial extension and size need not be physical reality but merely an idealized approximation from a mathematical standpoint. This idealization works very well in many cases but meets its limits with singularities. Actually, each quantum entity in nature may be a deformable medium of spatial shape and size. This idea was proposed by great pioneers in the quantum theory, e.g., P. A. M. DIRAC, W. HEISENBERG, W. PAULI, E. SCHRÖDINGER, and R. FEYNMAN. However, it has been found that the complexities and difficulties cannot be surmounted in various attempts to construct the quantum theory beyond the notion of the point particle.

Now, the situation may be changed by uncovering the correlation between the Schrödinger equation and the governing equations for deformable media. It has been demonstrated in the previous sections that a new formulation of the quantum theory may be established by replacing the notion of the point particle with the notion of the deformable medium. In the new formulation, the Schrödinger equation still plays a central and fundamental role in the sense of determining all dynamic variables. With the new results given by Eqs. (22)–(24) or (59)–(61), the new formulation based on the notion of the deformable medium is deterministic, accordingly, irrelevant to any probabilistic indeterminacy. Thus, the long-standing fundamental issues centered on the probabilistic indeterminacy may be unraveled with new interpretations.

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Appendix A Derivation of Eqs. (14) and (15)

Differentiating Eq. (12) with respect to t yields

$$\frac{\partial \rho}{\partial t} = 2\Big(R\frac{\partial R}{\partial t} + S\frac{\partial S}{\partial t}\Big). \tag{A1}$$

Equation (13) produces

$$\rho \boldsymbol{v} = \rho \frac{\hbar}{m}\nabla\Big(\arctan\Big(\frac{S}{R}\Big)\Big) = \frac{\hbar}{m}(R\nabla S - S\nabla R). \tag{A2}$$

Therefore,

$$\nabla \cdot (\rho \boldsymbol{v}) = \frac{\hbar}{m}(R\nabla^2 S - S\nabla^2 R). \tag{A3}$$

Then, substituting Eqs. (A1) and (A3) into the continuity equation (6), we can derive Eq. (14).

Consider Eq. (7). For the velocity field $\boldsymbol{v}$ given by Eq. (13), we have $\boldsymbol{v}\cdot(\nabla\boldsymbol{v}) = \frac{1}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v})$. Therefore, we may rewrite Eq. (7) as follows:

$$\nabla q - \rho\Big(\nabla U + m\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}m\nabla(\boldsymbol{v}\cdot\boldsymbol{v})\Big) = \mathbf{0}.$$

Then, using the equality $\rho\nabla A = \nabla(\rho A) - A\nabla\rho$, as well as Eq. (13), we may recast Eq. (7) as follows:

$$\nabla q - \nabla(\rho A) + A\nabla\rho = 0, \tag{A4}$$

where

$$A = U + \hbar\frac{\partial}{\partial t}\Big(\arctan\frac{S}{R}\Big) + \frac{\hbar^2}{2m}\Big|\nabla\Big(\arctan\frac{S}{R}\Big)\Big|^2. \tag{A5}$$

Applying Eq. (A2) and denoting

$$B = \rho U + \hbar\Big(R\frac{\partial S}{\partial t} - S\frac{\partial R}{\partial t}\Big) - \frac{\hbar^2}{2m}(R\nabla^2 R + S\nabla^2 S), \tag{A6}$$

$$C = R\nabla^2 R + S\nabla^2 S + \rho^{-1}|R\nabla S - S\nabla R|^2, \tag{A7}$$

we have

$$\rho A = B + \frac{\hbar^2}{2m}C.$$

Then, using the above equation and $\rho\nabla(\rho^{-1}B) = \nabla B - \rho^{-1}B\nabla\rho$, we can further reformulate Eq. (A4) as follows:

$$\rho\nabla(\rho^{-1}B) - \nabla q + \frac{\hbar^2}{2m}(\nabla C - \rho^{-1}(\nabla\rho)C) = \mathbf{0}. \tag{A8}$$

Now, utilizing the equalities (cf., Eq. (12))

$$R\nabla^2 R + S\nabla^2 S = \frac{1}{2}\nabla^2\rho - |\nabla R|^2 - |\nabla S|^2, \tag{A9}$$

$$|R\nabla S - S\nabla R|^2 = R^2|\nabla S|^2 + S^2|\nabla R|^2 - 2(R\nabla R)\cdot(S\nabla S), \tag{A10}$$

we can deduce

$$C = \frac{1}{2}\nabla^2\rho - \frac{1}{4}\rho^{-1}|\nabla\rho|^2, \tag{A11}$$

$$\rho^{-1}(\nabla\rho)C = \frac{1}{4}\nabla(\rho^{-1}|\nabla\rho|^2). \tag{A12}$$

Hence, we have

$$\frac{\hbar^2}{2m}(\nabla C - \rho^{-1}(\nabla\rho)C) = \frac{\hbar^2}{4m}\nabla(\rho\nabla^2(\ln\rho)). \tag{A13}$$

Thus, with Eqs. (12) and (13), the equation of motion, i.e., Eq. (7), may be converted to

$$\nabla(\rho^{-1}B) - \nabla q + \frac{\hbar^2}{4m}\nabla(\rho\nabla^2(\ln\rho)) = \mathbf{0}. \tag{A14}$$

From this and Eq. (10), we infer that Eq. (A8) may be reduced to $\nabla(\rho^{-1}B) = \mathbf{0}$. The last equation means that B should be constant. From this and Eq. (21), we infer that $B = 0$. Thus, from Eq. (A6), we can derive Eq. (15).

Appendix B Derivation of Eq. (39)

From Eq. (10) and the equality (cf., Eq. (6))

$$-\rho\nabla\cdot\boldsymbol{v}=\frac{\partial\rho}{\partial t}+\boldsymbol{v}\cdot\nabla\rho, \tag{B1}$$

we can recast Eq. (38) as follows:

$$W_0=-\frac{\hbar^2}{4m}\int_0^t\Big(\oint\nabla^2(\ln\rho)\Big(\frac{\partial\rho}{\partial t}+\boldsymbol{v}\cdot\nabla\rho\Big)\mathrm{d}V\Big)\mathrm{d}t.$$

The integrand in the above integration is given by the sum of the following two terms:

$$\begin{aligned}\nabla^2(\ln\rho)\frac{\partial\rho}{\partial t}&=\nabla\cdot\Big(\frac{\partial\rho}{\partial t}\nabla(\ln\rho)\Big)-(\nabla(\ln\rho))\cdot\nabla\Big(\frac{\partial\rho}{\partial t}\Big),\\ \nabla^2(\ln\rho)(\nabla\rho)\cdot\boldsymbol{v}&=\frac{1}{2}\nabla\cdot(|\nabla(\ln\rho)|^2\rho\boldsymbol{v})-\frac{1}{2}|\nabla(\ln\rho)|^2\nabla\cdot(\rho\boldsymbol{v})\\ &=\frac{1}{2}\nabla\cdot(|\nabla(\ln\rho)|^2\rho\boldsymbol{v})+\frac{1}{2}|\nabla(\ln\rho)|^2\frac{\partial\rho}{\partial t}.\end{aligned}$$

In deriving the latter term, Eq. (6) has been used. From the above two equalities and the equality

$$\frac{1}{2}|\nabla(\ln\rho)|^2\frac{\partial\rho}{\partial t}-(\nabla(\ln\rho))\cdot\nabla\Big(\frac{\partial\rho}{\partial t}\Big)=-\frac{1}{2}\frac{\partial}{\partial t}(\rho|\nabla(\ln\rho)|^2),$$

we can see that the foregoing integration may be given by

$$W_0=-\frac{\hbar^2}{8m}\int_0^t\Big(\oint\nabla\cdot(|\nabla(\ln\rho)|^2\rho\boldsymbol{v})\mathrm{d}V-\oint\frac{\partial}{\partial t}(\rho|\nabla(\ln\rho)|^2)\mathrm{d}V\Big)\mathrm{d}t.$$

From Gauss's theorem and Eq. (9), we can deduce that the first integration in the above equation vanishes. Let ρ be uniform at a natural state at $t=0$. Then, Eq. (39) may be derived.

Appendix C Derivation of Eq. (41)

Using Eq. (22) and (cf., Eq. (18))

$$\Psi^*\nabla^2\Psi=R\nabla^2R+S\nabla^2S+\mathrm{i}(R\nabla^2S-S\nabla^2R),$$

we may recast Eq. (40) as follows:

$$\langle H\rangle=-\frac{\hbar^2}{2m}\oint(R\nabla^2R+S\nabla^2S+\rho U)\mathrm{d}V+\frac{\mathrm{i}\hbar^2}{2m}\oint(S\nabla^2R-R\nabla^2S)\mathrm{d}V.$$

Applying Eq. (A9) for the first integration, utilizing the equality

$$S\nabla^2R-R\nabla^2S=\nabla\cdot(S\nabla R-R\nabla S)$$

for the second integration, and then applying Gauss's theorem with Eq. (21), we have

$$\langle H\rangle=\oint\Big(\frac{\hbar^2}{2m}(|\nabla R|^2+|\nabla S|^2)+\rho U\Big)\mathrm{d}V. \tag{C1}$$

From Eq. (A2), we can derive

$$\rho|\boldsymbol{v}|^2=\frac{\hbar^2}{m^2}(|\nabla R|^2+|\nabla S|^2-\frac{1}{4}\rho|\nabla(\ln\rho)|^2).$$

Therefore, we have

$$\oint\Big(\frac{1}{2}\rho|\boldsymbol{v}|^2+\rho U\Big)\mathrm{d}V=\frac{\hbar^2}{2m}\oint\Big(|\nabla R|^2+|\nabla S|^2+\rho U-\frac{\rho}{4}|\nabla(\ln\rho)|^2\Big)\mathrm{d}V.$$

From the above equation and Eqs. (37), (39), and (A16), we can infer that Eq. (41) holds.