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CRACK PROBLEM FOR AN INHOMOGENEOUS PLANE BONDED BY TWO DIFFERENT INHOMOGENEOUS HALF-PLANES*

Tang Ren-ji (汤任基)

(Shanghai Jiaotong University, Shanghai)

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Abstract

In this paper the crack problem for two bonded inhomogeneous half-planes is considered. It is assumed that the different materials have the same Poisson ratio ν *, but generally speaking, both Young's moduli vary exponentially with the coordinate* x *in different form. Using the single crack solution of the inhomogeneous plane problem and Fourier transform technique, the problem is reduced to a Cauchy-type singular integral equation. Several numerical examples to calculate the stress intensity factors are carried out.*

Key words inhomogeneous plane, crack problem, singular integral equation

I. Introduction

The crack problem in inhomogeneous media has been of considerable interests from the practical points of view. In fact, it is often encountered in geophysics and essentially inhomogeneous solids. Recently, Delale and Erdogan[1] have treated the crack problem for an inhomogeneous plane and a single crack solution is obtained. In this paper, using the crack solution and Fourier transform technique, the more complicated crack problem for two bonded inhomogeneous half-planes is further treated. In this case the stresses are obtained in terms of the unknown dislocation density defined on crack surface and then the problem is reduced to solve a Cauchy-type singular integral equation, from which we can prove that the stress state around the crack tips has the same square roote singularity as the case of homogeneous media. Finally several numerical examples are worked out for the cases of crack near the bonded line.

II. Two Basic Solutions

In order to utilize immediately the earlier results to study the crack problem for an inhomogeneous plane bonded by two different inhomogeneous half-plane, two basic solutions are given below

If Poisson ratio ν is a constant and Young's modulus is an exponential function as $E(x)=E_0\exp[\beta x]$, where E_0 and β are constants, then the stress function $F(x,y)$ of the plane problem of inhomogeneous elasticity satisfies the following equation which is a partial four-order differential equation

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$$\nabla^4 F-2\beta\left(\frac{\partial^3 F}{\partial x^3}+\frac{\partial^3 F}{\partial x \partial y^2}\right)+\beta^2\frac{\partial^2 F}{\partial x^2}-\nu\beta^2\frac{\partial^2 F}{\partial y^2}=0 \tag{2.1}$$

If the problem is symmetrical with respect to the axis Ox, equation (2.1) can be solved by use of the method of Fourier consine transform. Note that its general solution is given by

$$F(x,y)=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty \sum_{j=1}^{2} B_j(a)\exp[n_j x]\cos ay\,da & x\geqslant 0\\ \dfrac{2}{\pi}\displaystyle\int_0^\infty \sum_{j=3}^{4} B_j(a)\exp[n_j x]\cos ay\,da & x\leqslant 0\end{cases} \tag{2.2}$$

where

$$n_{1,3}=n_{1,3}(a,\beta)=(\beta\mp\sqrt{\beta^2+4a^2+4a\beta\sqrt{\nu}\,i})/2 \tag{2.3}$$

$$n_{2,4}=n_{2,4}(a,\beta)=(\beta\mp\sqrt{\beta^2+4a^2-4a\beta\sqrt{\nu}\,i})/2 \tag{2.4}$$

and $B_j(a)$ ($j=1,2,3,4$) are unknown functions which can be determined from the boundary conditions of the problem.

The result (2.2) is called Fourier cosine transform solution (i.e., the first basic solution).

If the straight crack (a,b) lies on real axis Ox as shown in Fig. 1, and the material parameters satisfy $\delta=\beta$, equation (2.1) can be solved by use of the Fourier transform and its general solution of the upper half-plane is given by

$$F^*(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}A_j(\rho)\exp[-m_j y]\exp[-i\rho x]d\rho$$

$$(-\infty<x<\infty,\ y>0) \tag{2.5}$$

In the case of symmetry about axis Ox, the unknown function $A_j(\rho)$ can be expressed by

$$A_1(\rho)=-\frac{m_2}{m_1}A_2(\rho)=\frac{E_0 m_1 m_2^2}{\rho^2(m_1^2-m_2^2)(\beta+i\rho)}$$

$$\cdot\int_a^b g(t)\exp[(\beta+i\rho)t]dt \tag{2.6}$$

where $g(t)$ is a following dislocation function of crack (a,b)

$$g(t)=\partial v(t,+0)/\partial t \qquad a<t<b \tag{2.7}$$

and

$$m_1=m_1(\rho,\beta)=(\beta\sqrt{\nu}+\sqrt{\beta^2\nu+4\rho^2-4\rho\beta i})/2=\overline{m_1(-\rho,\beta)} \tag{2.8}$$

$$m_2=m_2(\rho,\beta)=(-\beta\sqrt{\nu}+\sqrt{\beta^2\nu+4\rho^2-4\rho\beta i})/2=\overline{m_2(-\rho,\beta)} \tag{2.9}$$

The result (2.5) is called a single crack solution, the second basic solution, which is obtained firstly by F. Delale and F. Erdogan.

III. Integral Equation

Now we consider the crack problem for an inhomogeneous plane bonded by two different inhomogeneous half-planes as shown in Fig. 1. Assume that the different materials have the same Poisson ratio $\nu=$const., but Young's moduli take different exponential functions in regions Ω_1 and Ω_2 as follows

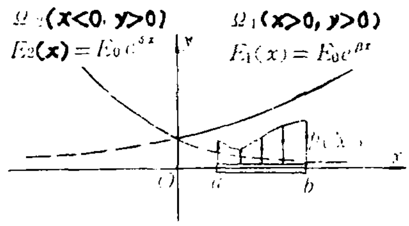

Fig. 1

$$E_1(x)=E_0\exp[\beta x] \qquad x\in\Omega_1 \tag{3.1}$$

$$E_2(x)=E_0\exp[\delta x] \qquad x\in\Omega_2 \tag{3.2}$$

The crack surfaces are loaded by symmetrical pressure $p(x)$, and the displacements remain continuous across the bonded line. Note that this problem can be solved by use of the above two basic solutions. From symmetry, it is sufficient to consider upper half-plane. It can be shown that the stress functions $F_1(x,y)$ in region Ω_1 and $F_2(x,y)$ in region Ω_2 can be expressed by the following forms

$$F_1(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}A_j(\rho)\exp[-m_jy]\exp[-i\rho x]d\rho$$

$$+\frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{2}B_j(a)\exp[n_jx]\cos ay da \qquad (x,y)\in\Omega_1 \tag{3.3}$$

$$F_2(x,y)=\frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{2}C_j(a)\exp[\lambda_jx]\cos ay da \qquad (x,y)\in\Omega_2 \tag{3.4}$$

where m_j and n_j are given by equations (2.3,4) and (2.8,9), the parameter $\lambda_j=\lambda_j(a,\delta)=n_{j+2}(a,\delta)$, the unknown functions $B_j(a)$ and $C_j(a)$ can be determined by the joining conditions of the bonded line.

The stresses and displacements in region Ω_1 can be calculated by use of the stress function $F_1(x,y)$ as follows

$$\sigma_{xx1}(x,y)=\frac{\partial^2F_1(x,y)}{\partial y^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}A_j(\rho)m_j^2\exp[-m_jy]\exp[-i\rho x]d\rho$$

$$-\frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{2}B_j(a)a^2\exp[n_jx]\cos ayda \tag{3.5}$$

$$\sigma_{yy1}(x,y)=\frac{\partial^2F_1(x,y)}{\partial x^2}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}A_j(\rho)\rho^2\exp[-m_jy]\exp[-i\rho x]d\rho$$

$$+\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 B_j(\alpha)n_j^2\exp[n_jx]\cos\alpha y d\alpha \tag{3.6}$$

$$\tau_{xy1}(x,y)=-\frac{\partial^2 F_1(x,y)}{\partial x\partial y}=-\frac{i}{2\pi}\int_{-\infty}^\infty \sum_{j=1}^2 A_j(\rho)\rho m_j\exp[-m_jy]\exp[-i\rho x]d\rho$$

$$+\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 B_j(\alpha)\alpha n_j\exp[n_jx]\sin\alpha y d\alpha \tag{3.7}$$

and

$$u_1(x,y)=-\frac{1}{2\pi}\int_{-\infty}^\infty \sum_{j=1}^2 \frac{A_j(\rho)(m_j^2+\nu\rho^2)\exp[-m_jy]\exp[-i\rho x]}{(\beta+i\rho)E_1(x)}d\rho$$

$$-\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 \frac{B_j(\alpha)(\alpha^2+\nu n_j^2)\exp[n_jx]\cos\alpha y}{(n_j-\beta)E_1(x)}d\alpha+D_1 \tag{3.8}$$

$$v_1(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty \sum_{j=1}^2 \frac{A_j(\rho)(\rho^2+\nu m_j^2)\exp[-m_jy]\exp[-i\rho x]}{m_jE_1(x)}d\rho$$

$$+\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 \frac{B_j(\alpha)(n_j^2+\nu\alpha^2)\exp[n_jx]\sin\alpha y}{\alpha E_1(x)}d\alpha \tag{3.9}$$

The stresses and displacements in region Ω_2 can be calculated by use of the stress function $F_2(x,y)$ as follows

$$\sigma_{xx2}(x,y)=\frac{\partial^2 F_2(x,y)}{\partial y^2}=-\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 C_j(\alpha)\alpha^2\exp[\lambda_jx]\cos\alpha y d\alpha \tag{3.10}$$

$$\sigma_{yy2}(x,y)=\frac{\partial^2 F_2(x,y)}{\partial x^2}=\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 C_j(\alpha)\lambda_j^2\exp[\lambda_jx]\cos\alpha y d\alpha \tag{3.11}$$

$$\tau_{xy2}(x,y)=-\frac{\partial^2 F_2(x,y)}{\partial x\partial y}=\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 C_j(\alpha)\alpha\lambda_j\exp[\lambda_jx]\sin\alpha y d\alpha \tag{3.12}$$

and

$$u_2(x,y)=-\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 \frac{C_j(\alpha)(\alpha^2+\nu\lambda_j^2)\exp[\lambda_jx]\cos\alpha y}{(\lambda_j-\delta)E_2(x)}d\alpha+D_2 \tag{3.13}$$

$$v_2(x,y)=\frac{2}{\pi}\int_0^\infty \sum_{j=1}^2 \frac{C_j(\alpha)(\lambda_j^2+\nu\alpha^2)\exp[\lambda_jx]\sin\alpha y}{\alpha E_2(x)}d\alpha \tag{3.14}$$

The constants D_1 and D_2 in equations (3.8) and (3.13) respectively are the rigid body displacements which can be removed by use of derivation.

Using following joining conditions of the bonded line Oy shown in Fig. 1.

$$\sigma_{xx1}(+0,y)=\sigma_{xx2}(-0,y), \qquad \tau_{xy1}(+0,y)=\tau_{xy2}(-0,y) \tag{3.15}$$

$$u_1(+0,y)=u_2(-0,y), \qquad v_1(+0,y)=v_2(-0,y) \tag{3.16}$$

then the unknown functions $B_j(\alpha)$ and $C_j(\alpha)$ can be determined. Back substituting them into equations (3.5 – 3.7) and (3.10 – 3.12), the useful stresses are obtained, where only an unknown dislocation function $g(t)$ is not yet determined. Further, let the stress $\sigma_{yy1}(x,y)$ in region Ω_1 satisfy the loading condition of the crack surface (a,b), i.e.

$$\lim_{y\to+0}\sigma_{yy1}(x,y)=-p(x) \qquad a<x<b \tag{3.17}$$

Then the integral equation of the above mentioned crack problem is derived as follows

$$\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x}+\mathscr{K}_1(x,0,t)+\mathscr{K}_2(x,0,t)\right]g(t)\exp[\beta t]dt=-\frac{1+\varkappa}{4\mu_0}p(x) \tag{3.18}$$

where $\varkappa=3-4\nu$ for plane strain, $\varkappa=(3-\nu)/(1+\nu)$ for plane stress, μ_0 is the shear modulus, the kernels are expressed by following bounded infinite integrals

$$\mathscr{K}_1(x,0,t)=\int_0^\infty\left[2\mathrm{Re}\frac{m_1m_2\exp[i\rho(t-x)]}{(m_1+m_2)(\beta+i\rho)}-\sin\alpha(t-x)\right]d\rho \tag{3.19}$$

$$\mathscr{K}_2(x,0,t)=\int_0^\infty\int_{-\infty}^\infty K_2(x,0,t;\ \alpha,\rho)d\rho d\alpha \tag{3.20}$$

and

$$K_2(x,0,t;\ \alpha,\rho)=\sum_{j=1}^4 K_{2j}(x,t;\ \alpha,\rho) \tag{3.21}$$

where

$$K_{21}(x,t;\ \alpha,\ \rho)=\frac{(n_2^2+\nu\alpha^2)n_1^2\exp[n_1x]-(n_1^2+\nu\alpha^2)n_2^2\exp[n_2x]}{n_2^2-n_1^2}$$

$$\cdot\frac{m_1^2m_2^2\exp[i\rho t]}{\pi\rho^2(\beta+i\rho)(m_1^2+\alpha^2)(m_2^2+\alpha^2)}$$

$$K_{22}(x,t;\ \alpha,\rho)=\frac{(-n_1^2\exp[n_1x]+n_2^2\exp[n_2x])\alpha^2}{n_2^2-n_1^2}$$

$$\cdot\frac{[(m_2^2+m_1^2)\rho^2+m_1^2m_2^2\nu+\alpha^2\rho^2]\exp[i\rho t]}{\pi\rho^2(\beta+i\rho)(m_1^2+\alpha^2)(m_2^2+\alpha^2)}$$

$$K_{23}(x,t;\ \alpha,\ \rho)=\frac{(n_1-\beta)(\alpha^2+\nu n_2^2)n_1^2\exp[n_1x]-(n_2-\beta)(\alpha^2+\nu n_1^2)n_2^2\exp[n_2x]}{(n_1-n_2)[(n_1+n_2)\alpha^2-\beta\alpha^2+n_1n_2\nu\beta]}$$

$$\cdot\frac{m_1^2m_2^2\exp[i\rho t]}{\pi i\rho(\beta+i\rho)(m_1^2+\alpha^2)(m_2^2+\alpha^2)}$$

$$K_{24}(x,t;\ \alpha,\ \rho)=\frac{[(n_1-\beta)(n_2-\beta)](n_2n_1^2\exp[n_1x]-n_1n_2^2\exp[n_2x])}{(n_1-n_2)[(n_1+n_2)\alpha^2-\beta\alpha^2+n_1n_2\nu\beta]}\cdot$$

$$\cdot\frac{m_1^2m_2^2(\alpha^2-\nu\rho^2)\exp[i\rho t]}{\pi\rho^2(\beta+i\rho)(m_1^2+\alpha^2)(m_2^2+\alpha^2)}$$

Note that, in the case of internal crack, the unknown dislocation function must satisfy the following condition of single-valuedness of the displacement.

$$\int_a^b g(t)dt=0 \tag{3.22}$$

Obviously, equation (3.18) is a Cauchy-type singular integral equation about unknown dislocation function $g(t)$, which can be solved by use of the numerical method[2] of singular integral equation. Having found the function $g(t)$ from this equation (3.18), the original crack problem is then solved.

IV. Stress Intensity Factor

Solving the above integral equation (3.18), we can obtain the solution $g(t)$ and substitute it in equation (3.6), then the stress $\sigma_{yy1}(x,y)$ in region Ω_1 is found out. In the case of internal crack, the dominant part of this stress in neighbourhood of the crack tips is

$$\sigma_{yy1}(x,0)=\frac{4\mu_0}{\pi(1+\varkappa)}\int_a^b\frac{g(t)\exp[\beta t]}{t-x}dt \tag{4.1}$$

so that the stress intensity factors of mode I can be determined by the well-known method as follows

$$k(a)=\lim_{x\to a}\sqrt{2(a-x)}\,\sigma_{yy1}(x,+0)=\frac{4\mu_0}{1+\varkappa}\lim_{x\to a}\sqrt{2(x-a)}\,g(x)\exp[\beta x] \tag{4.2}$$

$$k(b)=\lim_{x\to b}\sqrt{2(x-b)}\,\sigma_{yy1}(x,+0)=-\frac{4\mu_0}{1+\varkappa}\lim_{x\to b}\sqrt{2(b-x)}\,g(x)\exp[\beta x] \tag{4.3}$$

V. Numerical Results

In order to verify the method and illustrate its application, several numerical examples to calculate the stress intensity factors are carried out. Firstly, we consider the case where the crack half length $c=(b-a)/2$ and one of the material parameters (δ,β) are constants, but the crack center coordinate $d=(b+a)/2$ and the another parameter are varied. Then the results of the stress intensity factors for uniform crack surface pressure p_0 are listed in table 1. and table 2.

Table 1 Variation with parameters d and β of the stress intensity factors; $d=(a+b)/2$, $c=(b-a)/2=1$, $\delta=0$

d	1		3		5		$\to\infty$[1]	
βc	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$
1.0	0.703	1.172	0.735	1.194	0.737	1.196	0.740	1.209
0.5	0.817	1.089	0.856	1.107	0.859	1.109	0.861	1.113
0.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
−0.5	1.182	0.899	1.133	0.871	1.118	0.864	1.113	0.861
−1.0	1.412	0.788	1.223	0.743	1.200	0.738	1.197	0.738

Table 2 Variation with parameters d and δ of the stress intensity factors; $d=(a+b)/2$, $c=(b-a)/2=1$, $\beta=0$

d	1		3		5		$\to\infty$[1]	
δc	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$
1.0	1.110	1.062	1.017	1.013	1.007	1.006	1.000	1.000
0.5	1.048	1.035	1.011	1.009	1.005	1.005	1.000	1.000
0.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
−0.5	0.913	0.967	0.988	0.991	0.995	0.996	1.000	1.000
−1.0	0.868	0.948	0.983	0.987	0.993	0.994	1.000	1.000

Secondly, we consider the cases where the crack coordinates a and b keep constants, but δ and β are varied. The results of the stress intensity factors for uniform crack surface pressure p_0 are listed in table 3.

Table 3 Variation with material parameters (δ, β) of the stress intensity factors; $a=2$, $b=4$

δc	βc	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$	δc	βc	$k(a)/p_0\sqrt{c}$	$k(b)/p_0\sqrt{c}$
1.00	1.00	0.7354	1.1947	−1.00	1.00	0.7334	1.1918
0.75	0.75	0.7934	1.1537	−0.75	0.75	0.7899	1.1493
0.50	0.50	0.8585	1.1088	−0.50	0.50	0.8531	1.1028
0.25	0.25	0.9289	1.0584	−0.25	0.25	0.9224	1.0522
0.10	0.10	0.9717	1.0244	−0.10	0.10	0.9671	1.0204
0.00	0.00	1.0000	1.0000	0.00	0.00	1.0000	1.0000
−0.10	−0.10	1.0239	0.9722	0.10	−0.10	1.0305	0.9760
−0.25	−0.25	1.0581	0.9296	0.25	−0.25	1.0521	0.9156
−0.50	−0.50	1.1093	0.8598	0.50	−0.50	1.0367	0.8016
−0.75	−0.75	1.1551	0.7953	0.75	−0.75	1.0199	0.7250
−1.00	−1.00	1.1970	0.7377	1.00	−1.00	1.0283	0.6786

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